SPECTRAL THEORY FOR NON-NORMAL HILBERT SPACE OPERATORS

SAMEER CHAVAN

Abstract. In this talk, we discuss the spectral theory for some non-normal operators on separable Hilbert spaces. Recall that a Hilbert space $H$ is an inner-product space endowed with the inner-product $\langle \cdot, \cdot \rangle$, which is complete in the induced norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. By the Hilbert space operator $T$ on $H$, we mean a linear transformation $T: H \rightarrow H$ satisfying
$$\|Th\| \leq M\|h\| \quad (h \in H)$$
for some finite number $M > 0$. A linear operator $T^*: H \rightarrow H$ is said to be the adjoint of $T$ if it satisfies
$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad (x,y \in H).$$

A linear operator $N$ on $H$ is said to be normal if $N^*N = NN^*$. Multiplication operators on Hilbert spaces of square-integrable functions provide basic examples of normal operators. Any operator $T$ on $H$ for which $T^*T - TT^* \neq 0$ is non-normal. Any isometry (that is, an operator $T$ satisfying $T^*T = I$) which is not surjective is a rather special example of a non-normal operator.

In the first half of this talk, we discuss one of the cornerstones of the Operator Theory: Spectral Theorem for Normal Operators. This theorem roughly asserts that any normal operator $N$ can be obtained by integrating the co-ordinate function on the compact subset $\sigma(N)$ of the complex plane with respect to a “nice” projection-valued measure. Moreover, one can make sense out of $f(N)$ for a class of functions $f$ which include, in particular, continuous functions. In the remaining half, we address the following delicate question:

What is the “size” of $T^*T - TT^*$ for a non-normal operator $T$?

There are two well-studied classes of non-normal operators for which the above question can be answered to a greater extent. In the case of hyponormal operators (operators for which $T^*T - TT^* \geq 0$), the most remarkable answer is due to Berger and Shaw ([2]). In the case of 2-isometries (operators for which $I - 2T^*T + T^*2T^2 = 0$), the same is a recent achievement due to the speaker ([5]). Needless to say, these results have far-reaching consequences to the spectral theory for non-normal operators. The proof of the Berger-Shaw-type result for 2-isometries relies heavily on the basic theory of the so-called Cauchy dual operators, a subject of independent interest ([8]). We also plan to discuss some recent developments related to the Cauchy dual operators in the unbounded and multi-variable operator theory ([6] and [7]).

References


SAMEER CHAVAN
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Kanpur- 208016
E-mail Address: chavan@iitk.ac.in