1. Adic spaces

For any Huber pair \((A, A^+)\) the space \(X = \text{Spa}(A, A^+)\) has a pre-sheaf of complete topological rings \(\mathcal{O}_X\). The Huber pair is called sheafy if \(\mathcal{O}_X\) is a sheaf. Moreover for any \(x \in X\) the stalk \(\mathcal{O}_{X,x}\) is a local ring equipped with a valuation \(v_x : \mathcal{O}_{X,x} \to \Gamma_x \cup \{0\}\).

Let \(\mathcal{V}\) be the category whose objects are triples \(\mathcal{X} = (X, \mathcal{O}_X, (v_x)_{x \in X})\), where

- \(X\) is a topological space,
- \(\mathcal{O}_X\) is a sheaf of complete topological rings, which makes \(X\) a locally ringed space
- \(v_x : \mathcal{O}_{X,x} \to \Gamma_x \cup \{0\}\) is a continuous valuation.

A morphism in \(\mathcal{V}\), \(f : \mathcal{X} \to \mathcal{Y}\) is a morphism of locally ringed spaces \(f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) such that the maps \(\mathcal{O}_Y(f^{-1}U) \to \mathcal{O}_X(U)\) are continuous for ever open \(U \subseteq X\) and there exist order preserving homomorphisms of abelian groups \(\Gamma_{f(x)} \to \Gamma_x\) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\
\downarrow & & \downarrow \\
\Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\}
\end{array}
\]

upto equivalence.

An adic space is an object \(\mathcal{X}\) in \(\mathcal{V}\) such that \(X = \bigcup U_i\) where \(U_i, \mathcal{O}_X|_{U_i}, (v_x)_{x \in U_i}\) is isomorphic to \(\text{Spa}(A_i, A_i^+)\) for a sheafy Huber pair \((A_i, A_i^+)\) in \(\mathcal{V}\). Morphisms of adic spaces are morphisms in \(\mathcal{V}\).

For a sheafy Huber pair the triple \((X = \text{Spa}(A, A^+), \mathcal{O}_X, (v_x)_{x \in X})\) is called an affinoid adic space. We denote by \(\mathcal{A}d\) the category of adic spaces.

2. The functor of points for adic spaces

Just like schemes any adic space \(X\) gives a functor \(\mathcal{A}d \to \text{Sets}\), called its functor of points. If \(Y\) is another adic space, then

\[X(Y) = \text{Hom}_{\mathcal{A}d}(Y, X)\]

If \(Y = \text{Spa}(A, A^+)\), then \(X(Y) = \text{Hom}_{\text{Hub}}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))\) in the category of Huber pairs. Hence \((A, A^+) \to \text{Spa}(A, A^+)\) is a fully faithful functor from the category of sheafy Huber pairs to adic spaces.

3. Examples of Adic spaces

3.1. The final object. \((A, A)\) is a Huber pair for any discrete ring \(A\). Consider the ring \(\mathbb{Z}\).

The final object in \(\mathcal{A}d\) is \(\text{Spa}(\mathbb{Z}, \mathbb{Z})\). This space has 3 types of points:

1. \(\eta\) corresponding trivial valuation on \(\mathbb{Z}\).
2. The points \(s_p\) corresponding to the pull back of the trivial valuation on \(\mathbb{F}_p\) by the quotient map \(\mathbb{Z} \to \mathbb{F}_p\) for each prime \(p \in \mathbb{Z}\).
3. The points \(\eta_p\) corresponding to the \(p\)-adic valuation, \(|n| = p^{-\alpha}\) if \(n = p^\alpha m\) where \(p \nmid m\) and \(|0| = 0\).

It is easy to see that \(\eta\) is open while the points \(s_p\) are closed. On the other hand, \(\eta = \text{Spa}(\mathbb{Z}, \mathbb{Z})\) and \(\mathbb{Z}^\circ = \{\eta_p, s_p\}\).

There is a unique map from \((\mathbb{Z}, \mathbb{Z})\) to any Huber pair \((A, A^+)\), hence a unique map \(\text{Spa}(A, A^+) \to \text{Spa}(\mathbb{Z}, \mathbb{Z})\).

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This space can be represented by the following diagram

\[
\begin{array}{c}
\eta \\
\eta_2 \quad \eta_3 \quad \eta_5 \\
| \quad | \quad | \\
\eta_7 \quad \eta_9 \\
| \quad | \quad | \\
\eta_{11} \\
\cdots
\end{array}
\]

On the other hand if \( K \) is a non-archimedean field with valuation ring \( K^o \) then \( \text{Spa}(K, K^o) \) has a single point. An adic space over \( K \) is an adic space \( X \) with a morphism to \( \text{Spa}(K, K^o) \). A morphism between adic spaces over \( K \) is a morphism \( X \to Y \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
\text{Spa}(K, K^o) & \to & \text{Spa}(K, K^o)
\end{array}
\]

The category of adic spaces over \( K \) will be denoted by \( \mathcal{A}d/K \), in which the final object clearly is \( \text{Spa}(K, K^o) \).

3.2. **Closed unit disc.** Over \( \mathbb{Z} \) the closed unit disc is \( D_\mathbb{Z} = \text{Spa}(\mathbb{Z}[T], \mathbb{Z}[T]) \), where \( \mathbb{Z}[T] \) is discrete. Note that this is justified by its functor of points \( D_\mathbb{Z}(\text{Spa}(A, A^+) = A^+ \).

The closed adic unit disc over \( \mathbb{Q}_p \) is the affinoid adic space

\[
D_{\mathbb{Q}_p} := \text{Spa}(\mathbb{Q}_p(T), \mathbb{Z}_p(T)).
\]

Here the topology on \( \mathbb{Q}_p(T) \) comes from the sup norm

\[
\sum_{n=0}^{\infty} a_n T^n = \sup \{|a_n| : n \geq 0\}.
\]

Note that for any \( |\cdot| \in D_{\mathbb{Q}_p}, |T| \leq 1 \). Moreover for any point \( \alpha \in \overline{\mathbb{Q}_p}, |\alpha| \leq 1 \) there is a valuation \( \cdot |_{\alpha} = D_{\mathbb{Q}_p} \), given by \( |f|_{\alpha} = |f(\alpha)| \). Thus

\[
D_{\mathbb{Q}_p} \supset \{\alpha \in \overline{\mathbb{Q}_p} : |\alpha| \leq 1\}/\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p).
\]

However \( D_{\mathbb{Q}_p} \) has many more points. For example if

\[
B(\alpha, r) = \{\beta \in \overline{\mathbb{Q}_p} : |\beta - \alpha| \leq r\}
\]

is the closed ball of radius \( r \) around \( \alpha \), where \( \alpha \in \overline{\mathbb{Q}_p} \) with \( |\alpha| \leq 1 \) and \( 0 < r \leq 1 \), then there is a point in \( D_{\mathbb{Q}_p} \) corresponding to \( B(\alpha, r) \) given by the valuation

\[
|f| = \sup \{|f(\beta)| : \beta \in B(\alpha, r)\}.
\]

This is called the Gauss point of \( B(\alpha, r) \).

There is a natural map \( D_{\mathbb{Q}_p} \to \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \). If \( \text{Spa}(A, A^+) \) is an adic space over \( \mathbb{Q}_p \) then

\[
\text{Hom}_{\mathcal{A}d/\mathbb{Q}_p} \left( \text{Spa}(A, A^+), D_{\mathbb{Q}_p}^{\mathcal{A}d} \right) = A^+.
\]

There is a nice classification of points of \( D_K \) which can be found in Scholze’s paper on perfectoid spaces in Publ. IHES, for an algebraically closed non-archimedean field \( K \).

3.3. **Open Unit disc.** Over \( \mathbb{Z} \) the open unit disc is \( D_{\mathbb{Z}} = \text{Spa}(\mathbb{Z}[[T]], \mathbb{Z}[[T]]) \) where we give \( \mathbb{Z}[[T]] \) the \( T \)-adic topology. Let \( (A, A^+) \) be a complete Huber pair, it can be shown that

\[
D_{\mathbb{Z}}(\text{Spa}(A, A^+)) = A^{\infty}
\]

where \( A^{\infty} \) is the ideal of topologically nilpotent elements. Since \( T \) is topologically nilpotent in \( \mathbb{Z}[[T]] \) it has to go to a topologically nilpotent element in \( A \). Conversely sending \( T \) to any topologically nilpotent element of \( A \) gives a continuous ring homomorphism \( \mathbb{Z}[[T]] \to A \).

The open unit disc over \( \mathbb{Q}_p \) is a bit harder to define. Consider \( \mathbb{Z}_p[[T]] \) with \((p,T)\)-adic topology and let \( X = \text{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) \). There is a natural map \( X \to \text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) \). There are exactly two points in \( \text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) \),

- the point \( \eta \) corresponding to the \( p \)-adic valuation obtained from the inclusion \( \mathbb{Z}_p \to \mathbb{Q}_p \),
• the closed point \(s\) corresponding to the pull-back of the trivial valuation on \(F_p\) by the map \(\mathbb{Z}_p \to F_p\).

The closure of \(\eta\) is the entire space so \(\eta\) is the generic point. Moreover \(\{\eta\}\) is the rational open set \(\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)(\mathbb{p}/\mathbb{p})\) and so is isomorphic to \(\text{Spa}(\mathbb{Q}, \mathbb{Z}_p)\).

The generic fiber \(X_{\eta}\) with its natural map to \(\text{Spa}(\mathbb{Q}, \mathbb{Z}_p)\), is the open unit disc over \(\mathbb{Q}_p\) which we denote \(D^o_{\mathbb{Q}_p}\).

If \(|\cdot| \in D^o_{\mathbb{Q}_p}\), then since \(T\) is topologically nilpotent, we have \(|T^n| \to 0\), hence there is a \(k > 0\) such that \(|T^k| \leq |p| \neq 0\), thus \(|\cdot| \in X(T^k/p)\). Hence

\[
D^o_{\mathbb{Q}_p} = \bigcup_{k \geq 1} X(T^k/p).
\]

This is an open cover which does not have a finite sub-cover, thus \(D^o_{\mathbb{Q}_p}\) is not quasi-compact and hence not affinoid.

**Exercise.** Show that \(\text{Hom}_{\text{Ad}/\mathbb{Q}_p}(\text{Spa}(A, A^+), D^o_{\mathbb{Q}_p}) = \tilde{A}^o\).

### 3.4. Affine Line

The affine line over \(\mathbb{Z}\) is \(\mathbb{A}^1_\mathbb{Z} = \text{Spa}(\mathbb{Z}[T], \mathbb{Z})\), where \(\mathbb{Z}[T]\) has discrete topology. Clearly

\[
\mathbb{A}^1_\mathbb{Z}(\text{Spa}(A, A^+)) = A.
\]

Over \(\mathbb{Q}_p\) the affine line is given by a union over closed discs of increasing radii. Consider the following inclusion of Huber pairs

\[
\cdots \subset (\mathbb{Q}_p \langle p^2 T \rangle, \mathbb{Z}_p \langle p^2 T \rangle) \subset (\mathbb{Q}_p \langle pT \rangle, \mathbb{Z}_p \langle pT \rangle) \subset (\mathbb{Q}_p \langle T \rangle, \mathbb{Z}_p \langle T \rangle).
\]

This gives successive embeddings of adic spaces and we can take the union

\[
\mathbb{A}^1_{\mathbb{Q}_p} = \bigcup_{n=0}^{\infty} \text{Spa}(\mathbb{Q}_p \langle p^n T \rangle, \mathbb{Z}_p \langle p^n T \rangle)
\]

which is manifestly an adic space that is not quasi-compact and hence not affinoid.

In fact \(\text{Spa}(\mathbb{Q}_p \langle p^n T \rangle, \mathbb{Z}_p \langle p^n T \rangle)\) is the closed disc of radius \(1/|p|^n\), since \(|p^n T| \leq 1 \Rightarrow |T| \leq 1/|p|^n\).

**Exercise.** Show that \(\text{Hom}_{\text{Ad}/\mathbb{Q}_p}(\text{Spa}(A, A^+), \mathbb{A}^1_{\mathbb{Q}_p}) = A\).

### 3.5. Projective Line

The projective line over \(\mathbb{Z}\) can be constructed by gluing two copies of \(\mathbb{A}^1_\mathbb{Z}\), which I leave as an exercise.

In case of \(\mathbb{Q}_p\), consider the the closed unit disc \(D^o_{\mathbb{Q}_p}\) and take the unit circle

\[
S^1_{\mathbb{Q}_p} = \{|\cdot| \in D^o_{\mathbb{Q}_p} : 1 = |T|\} = D^o_{\mathbb{Q}_p}(1/T).
\]

This is a rational open subset isomorphic to \(\text{Spa}(\mathbb{Q}_p \langle T, T^{-1} \rangle, \mathbb{Z}_p \langle T, T^{-1} \rangle)\). The projective line is obtained by gluing two copies of the closed unit disc along the unit circle

\[
\mathbb{P}^1_{\mathbb{Q}_p} = \text{Spa}(\mathbb{Q}_p \langle T_1 \rangle, \mathbb{Z}_p \langle T_1 \rangle) \sqcup \text{Spa}(\mathbb{Q}_p \langle T_2 \rangle, \mathbb{Z}_p \langle T_2 \rangle)/\sim,
\]

where the identification \(\sim\) is obtained by the isomorphism

\[
\phi : (\mathbb{Q}_p \langle T_1, T_1^{-1} \rangle, \mathbb{Z}_p \langle T_1, T_1^{-1} \rangle) \to (\mathbb{Q}_p \langle T_2, T_2^{-1} \rangle, \mathbb{Z}_p \langle T_2, T_2^{-1} \rangle)
\]

given by \(\phi(T_1) = T_2^{-1}\).

Hence clearly \(\mathbb{P}^1_{\mathbb{Q}_p}\) is quasi-compact, but it is again not affinoid. To see this let us investigate the ring of global sections of the structure sheaf. If \(\sigma \in \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}(\mathbb{P}^1_{\mathbb{Q}_p})\) then \(\sigma = (\sigma_1, \sigma_2)\) where \(\sigma_i \in \mathbb{Q}_p \langle T_i \rangle\) such that \(\phi(s_1) = s_2\). This forces \(s_i\) to be constants and \(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}(\mathbb{P}^1_{\mathbb{Q}_p}) = \mathbb{Q}_p\) with \(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}^+(\mathbb{P}^1_{\mathbb{Q}_p}) = \mathbb{Z}_p\). Of course \(\mathbb{P}^1_{\mathbb{Q}_p}\) is not \(\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)\).
4. SCHEMES, FORMAL SCHEMES AND RIGID SPACES AS ADIC SPACES

4.1. Schemes. If $A$ is a discrete ring then $(A, A)$ is a sheafy Huber pair and there are maps of locally ringed spaces

$$\text{Spec } A \to \text{Spa}(A, A) \to \text{Spec } A.$$ 

The first map is given by sending a prime ideal $P \subset A$ to the trivial valuation on $A/P$, where as the second map is obtained by taking the support of a valuation. Clearly the composition is identity.

There is a fully faithful functor from the category of schemes to $\mathcal{A}$ which sends $\text{Spec } A$ to $\text{Spa}(A, A)$ for a discrete ring $A$.

4.2. Formal Schemes. If $A$ is an adic ring (it is a complete topological ring with $I$-adic topology for some ideal $I$) then $(A, A)$ is again a sheafy Huber pair. The formal scheme associated to $A$ is a locally ringed space of topologically complete rings, denoted by $\text{Spf}(A)$.

Again there is a fully faithful functor from formal schemes to adic spaces sending $\text{Spf}(A)$ to $\text{Spa}(A, A)$.

4.3. Rigid spaces. Let $K$ be a algebraically closed non-archimedian field, (think of $\mathbb{C}_p$), then an affinoid $K$-algebra is a complete normed $K$-algebra $A$ which is a quotient of $K\langle T_1, \ldots, T_n \rangle$ for some $n$. The rigid space associated to $A$ is a locally ringed Grothendieck-topologised space whose underlying set is

$$\text{Spm}(A) = \{m \subset A \mid m \text{ maximal ideal } \}.$$ 

This space is given a Grothendieck topology and has a structure sheaf with respect to that topology.

The associated adic space to $\text{Spm}(A)$ is $\text{Spa}(A, A^o)$. This extends to a fully faithful functor from the category of rigid spaces over $K$ to $\mathcal{A}/K$.

As an example consider the closed unit poly-disc $\text{Spm}(K\langle T_1, \ldots, T_n \rangle)$; the associated adic space is the closed unit poly disc $\text{Spa}(K\langle T_1, \ldots, T_n \rangle, K^o\langle T_1, \ldots, T_n \rangle)$.