Maryam Mirzakhani memorial talk
IISER Pune, Maths Club

November 17, 2017
Short Bio

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She did her graduate studies at Harvard University, under the supervision of Fields medallist Curtis T. McMullen and obtained her PhD in 2004.
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Geodesics on hyperbolic surfaces

In her PhD thesis, Mirzakhani studies the growth of the number $s_X(L)$ of simple closed geodesics of length at most $L$ on a closed hyperbolic surface $X$. Using this she gave a striking new proof of the Witten conjecture. The first proof of this conjecture was given by Maxim Kontsevich. This conjecture has deep consequences in quantum gravity, a field of theoretical physics, that seeks to describe gravity according to the principles of quantum mechanics.
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Hyperbolic plane

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Angles: Same as Euclidean angles.
Hyperbolic plane
Hyperbolic Geometry

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3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

However, it does not satisfy the 5th postulate, or the parallel postulate:

5. Given any straight line and a point not on it, there exists one and only one straight line which passes through that point and never intersects the first line.
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Infinite Parallels
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Hyperbolic distance

Recall in Euclidean plane if \( \gamma : [0, 1] \to \mathbb{R}^2 \) is a curve, and \( \gamma(t) = (x(t), y(t)) \) then its length is

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\ell(\gamma) = \int_0^1 |\gamma'(t)|\,dt = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.
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In the hyperbolic plane we measure lengths of curves differently. If $\gamma : [0, 1] \to D, \gamma(t) = (x(t), y(t))$ then length of $\gamma$ is

$$\ell(\gamma) = \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt = \int_0^1 \frac{2\sqrt{(x'(t))^2 + (y'(t))^2}}{1 - x^2(t) - y^2(t)} dt.$$
It turns out with this length measure also called \textit{hyperbolic metric}, the shortest curve between any two points is the unique circle passing through those points and meeting the boundary at right angles.
Geodesics

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The distance between any two points in the Hyperbolic plane is the length of the shortest curve joining the two points. Hence the length of the unique geodesic between those points.
Geodesics
Distance

Geodesic between the origin $O = (0, 0)$ and the point $A = (a, 0)$ in $D$ is the straight line $OA$, parametrized by

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\gamma : [0, 1] \rightarrow D, \quad \gamma(t) = (0, at).
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Note that $d(O, A) \rightarrow \infty$ as $a \rightarrow 1$.

All distances can be calculated using this, since there are isometries of $D$ that take any two points to the origin and a point on the $x$-axis.
Hyperbolic geometry

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- $D$ is a **Riemannian manifold** of dimension 2, since it is an (open) subset of $\mathbb{R}^2$ and we can measure lengths of curves.

- The geometry of $D$ is a type of **non-euclidean geometry** since it does not satisfy the parallel postulate of Euclid.
Closed Hyperbolic surfaces

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These manifolds arise naturally in nature and also in physics. For instance there is something called the world sheet in String theory. It is the 2 dimensional manifold traced out by a string moving in space and can be realised as a hyperbolic surface in certain cases.
The genus of a closed surface is just the number of holes it has. A surface is hyperbolic if it has genus at least 2.

Genus 2

Genus 3
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Moduli Space

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Points of $\mathcal{M}_g$ correspond to different hyperbolic surfaces. For two different points the corresponding surfaces are not isomorphic.

$\mathcal{M}_g$ is called the moduli space of genus $g$ hyperbolic surfaces and this is a major topic of study in mathematics, investigated by numerous mathematicians like Riemann, Mumford, Deligne, Kontsevic, Okounkov to name a few.
Simple closed geodesic

Let $X$ be a closed hyperbolic surface. A path $\gamma : [0, 1] \to X$ is a simple closed geodesic if:

1. $\gamma([0, 1])$ is a geodesic,
2. $\gamma(0) = \gamma(1),$
3. $\gamma(s) \neq \gamma(t)$ if $0 \leq s < t < 1.$

In words, $\gamma$ has the same starting and ending points, which is also a geodesic and which does not cross itself.
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In words $\gamma$ has the same starting and ending points, which is also a geodesic and which does not cross itself.
Curves

Red curve is not closed, blue curve is closed but not simple, green curve is simple and closed.
Mirzakhani’s result

Let us now fix a closed hyperbolic surface of genus $g$, $X \in \mathcal{M}_g$. 

Let $s_X(L)$ be the number of simple closed geodesics in $X$ whose length is at most $L$. Then Mirzakhani proves that asymptotically $s_X(L) \sim \eta(X)L^{6g-6}$ where $\eta(X)$ is a constant depending on the surface $X$. Moreover $\eta : \mathcal{M}_g \to \mathbb{R}^+$ is a continuous function.
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