To Maa

Abstract

The Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ Re(s) > 1.$$

This thesis presents a new and exceedingly simple method of analytic continuation of $\zeta(s)$ to the whole complex plane. This method involves an application of the binomial theorem. As a consequence, we give a simple inductive proof of the well known formula

$$\zeta(1-k) = \ -B_k/k,$$

where B_k is the k-th Bernoulli number. This technique is capable of wider generalization. We will consider the series

$$\zeta(s, z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$$

where z is a complex number in a much larger domain:

$$\mathbb{D}' = \{z \in \mathbb{C}\} \setminus \{z \in \mathbb{R} : z \leq 0\}$$

We will derive the analytic continuation for z in the larger complex domain by the method mentioned above and thereby extend the classical result

$$\zeta(1-k, a) = -\frac{B_k(a)}{k}$$

for $0 < a < 1$ to all $a \in \mathbb{D}'$, where $B_k(x)$ denotes the k-th Bernoulli polynomial. More generally, we will consider series of the type

$$\sum_{n=0}^{\infty}(n+z_n)^{-s}, \{z_n\} \text{ being a sequence in } \mathbb{D},$$

where $\mathbb{D}$ is the cut unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \setminus \{z \in \mathbb{R} : -1 < z \leq 0\} .$$

We will also consider poly-Hurwitz zeta functions defined as

$$\zeta(s_1, s_2, \dots, s_r; x_1, x_2, \dots, x_r) = \sum_{n=0}^{\infty} (n + x_1)^{-s_1} \dots (n + x_r)^{-s_r},$$

for $0 < x_i \leq 1$. The above series absolutely converges for $Re(s_1 + \cdots + s_r) > 1$. Using methods similar to the one mentioned above, we will be able to obtain a meromorphic continuation to $\mathbb{C}^r$.

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Contents

Chapter 1

Introduction

The Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is any complex variable, satisfying $Re(s) > 1$. This function was studied by Riemann in 1859. He wrote a paper in which he described the ζ function as the main tool to prove the prime number theorem. He observes that the above series is connected to prime numbers via the Euler product:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is over all primes p.

In the same paper, Riemann shows that $\zeta(s)$ admits an analytic continuation for all $s \in \mathbb{C}$ apart from $s = 1$, where it has a simple pole. He also derives the functional equation

$$\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-\frac{s}{2}} = \Gamma\left(\frac{(1-s)}{2}\right)\zeta(1-s)\pi^{-\frac{(1-s)}{2}}.$$

The proof presented in this thesis is an application of the Poisson Summation formula, and closely follows Riemann. I will follow the proof as treated in [M2].

In the third chapter of this thesis, we will study the analytic continuation of the Riemann zeta function. We will use a simple and elegant summation technique to analytically continue the ζ function. I have followed the treatment as outlined in [MR].

One of the main contributions of this thesis is a new and exceedingly simple method for the analytic continuation of $\zeta(s)$. This method involves an application of the binomial theorem. For this proof, we will also need the Hurwitz Zeta function, which is defined as

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

where x is a real number such that $0 < x \leq 1$. This series converges absolutely for $Re(s) > 1$. In this thesis, we present a new method of analytic continuation for $\zeta(s,x)$.

We will also give a new proof of the identity

$$\zeta(1-k) = -\frac{B_k}{k},$$

where the Bernoulli numbers B_n are defined by the power series

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

In the fourth chapter, we will introduce the Hurwitz zeta function

$$\zeta(s,x),\ x \in \mathbb{R},\ 0 < x \leq 1,$$

first studied by Hurwitz in 1882. In his paper, he obtained an analytic continuation of $\zeta(s,x)$ to the whole complex plane apart from $s = 1$, where it has a simple pole. Hurwitz thus unified the treatment of $\zeta(s)$ and the study of the classical Dirichlet L-functions $L(s,\chi)$ by observing that

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, \frac{r}{k}),$$

where χ is any Dirichlet character mod k and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},\ Re(s) > 1.$$

In the fourth chapter, we will study $\zeta(s,z)$ for a complex number z in domain $\mathbb{D}$, where $\mathbb{D}$ is the cut unit disc

$$\mathbb{D} = \{z : |z| < 1\} \setminus \{z \in \mathbb{R} : -1 < z \leq 0\}.$$

We will be able to generalise our new method for analytically continuing $\zeta(s)$ to obtain an analytic continuation of $\zeta(s,z)$ by applying the binomial theorem.

This will enable us to prove that

$$\zeta(1-k,z) = -\frac{B_k(z)}{k}, \tag{1.1}$$

where $B_k(z)$, the k-th Bernoulli polynomial is defined as

$$B_k(z) = \sum_{j=0}^{k} \binom{k}{j} B_j z^{k-j}.$$

This idea originates in the paper[MR], albeit for real values of $z, 0 < z \leq 1$. By a slight modification to our arguments, we can also define $\zeta(s,z)$ for z belonging to an even larger domain

$$\mathbb{D}' = \{z \in \mathbb{C}\} \setminus \{z \in \mathbb{R} : z \leq 0\}.$$

It will be shown that $\zeta(s,z)$, $z \in \mathbb{D}'$ can be analytically continued to the whole complex plane except $s = 1$ where it has a simple pole. Furthermore, equation (1.1) holds for all $z \in \mathbb{D}'$.

In the last chapter, we will define the Generalised Hurwitz zeta series

$$\zeta(s,\Lambda) = \sum_{n=0}^{\infty} \frac{1}{(n+z_n)^s},$$

where Λ is a sequence $\{z_n\}, z_n \in \mathbb{D}$, and study some applications to regularized products.

We also consider poly-Hurwitz zeta functions defined as

$$\zeta(s_1, s_2, \ldots, s_r; x_1, x_2, \ldots, x_r) = \sum_{n=0}^{\infty} (n+x_1)^{-s_1} \ldots (n+x_r)^{-s_r},$$

for $0 < x_i \leq 1$. By the same methods, we will obtain analytic continuation for these functions and prove as a consequence that

$$\zeta(1-k_1, 1-k_2, \ldots, 1-k_r; x_1, \ldots, x_r)$$

is a polynomial in $x_1, x_2, \ldots, x_r$ with rational coefficients (given by Bernoulli numbers). We may also consider these zeta functions for $x_i \in \mathbb{D}$ and thereby get a generalization of equation (1.1).

Chapter 2

Preliminaries

2.1 Objectives

The aim of this chapter is to introduce some fundamental definitions and results which will be required to prove the main results presented in this thesis.

2.2 The binomial theorem for complex exponents

For the sake of completeness, we first review the binomial theorem, which states that: [LA], page 59

Lemma 2.2.1 *If* $|z| < 1$ *and* α *is any complex number, then*

$$(1+z)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} z^r.$$

Here,

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!},\ \alpha \in \mathbb{C}$$

This lemma will be applied to prove the analytic continuation of the zeta function to the whole complex plane in chapter 3.

2.3 The Euler summation formula

The following result is very useful, and is referred to as the technique of partial summation.

Theorem 2.3.1 *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of complex numbers and* $f(t)$ *is a continuously differentiable function on* $[1,x]$. *Set* $A(t)=\sum_{n\le t} a_n$. *Then,*

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. Suppose first that x is a natural number.

Then,

$$\begin{aligned}\sum_{n\le x} a_n f(n) &= \sum_{n\le x} A(n)f(n) - \sum_{n\le x} A(n-1)f(n) \\ &= \sum_{n\le x} A(n)f(n) - \sum_{n\le x-1} A(n)f(n+1) \\ &= A(x)f(x) - \sum_{n\le(x-1)} A(n)\int_n^{n+1} f'(t)dt \\ &= A(x)f(x) - \sum_{n\le(x-1)} \int_n^{n+1} A(t)f'(t)dt,\end{aligned}$$

since $A(t)$ is a step function.

We observe

$$\sum_{n\le(x-1)} \int_n^{n+1} A(t)f'(t)dt = \int_1^x A(t)f'(t)dt.$$

Thus, the result has been proved if x in an integer. If x is not an integer, an observation that

$$A(x)\{f(x)-f([x])\} - \int_{[x]}^x A(t)f'(t)dt = 0$$

completes the proof. □

Corollary 2.3.2 *Suppose $A(x) = O(x^\delta)$. Then, for $Re(s) > \delta$,*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Hence, the Dirichlet series converges for $Re(s) > \delta$.

Proof. By Theorem 2.3.1, taking $f(n) = n^{-s}$, we get,

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(t)}{t^{s+1}} dt.$$

For s fixed, $Re(s) > \delta$, we have,

$$\lim_{x \to \infty} \frac{A(x)}{x^s} = 0$$

as we know that $A(x) = O(x^\delta)$. Thus,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

□

In the special case $a_n = 1$, it follows that

Corollary 2.3.3 *For $Re(s) > 0$,*

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\} = x - [x]$.

Proof. From above,

$$\sum_{n \leq x} \frac{1}{n^s} = s \int_1^x \frac{[t]}{t^{s+1}} dt.$$

Thus,

$$\begin{aligned} \zeta(s) &= s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx \\ &= s \int_1^{\infty} \frac{1}{x^s} dx - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx \\ &= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx, \end{aligned}$$

and the latter integral converges for $Re(s) > 0$. □

2.4 Poisson Summation formula

Theorem 2.4.1 *Let $F \in L^1(\mathbb{R})$. Suppose that the series $\sum_{n\in\mathbb{Z}} F(n+v)$ converges absolutely and uniformly in v, and that $\sum_{m\in\mathbb{Z}} |\widehat{F}(m)| < \infty$, where $\widehat{F}(m) = \int_{-\infty}^{\infty} F(x)e^{-2\pi imx}$. Then,*

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{2\pi inv}.$$

The following proof closely follows the treatment in [M2].
Proof. The function

$$G(v) = \sum_{n\in\mathbb{Z}} F(n+v)$$

is bounded, measurable and periodic with period 1.

The Fourier coefficients of G are

$$\begin{aligned} c_m &= \int_0^1 G(v)e^{-2\pi imv}dv \\ &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} F(n+v)\right) e^{-2\pi imv}dv \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 F(n+v)e^{-2\pi imv}dv \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} F(x)e^{-2\pi imx}dx \\ &= \int_{-\infty}^{\infty} F(x)e^{-2\pi imx}dx \\ &= \widehat{F}(m). \end{aligned}$$

Since

$$\sum_{n\in\mathbb{Z}} |\widehat{F}(m)| < \infty,$$

we can represent G by its Fourier series

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{2\pi inv}.$$

□

Corollary 2.4.2 *With F as above,*

$$\sum_{n\in\mathbb{Z}} F(n) = \sum_{n\in\mathbb{Z}} \widehat{F}(n).$$

Proof. Set $v = 0$ in Theorem (2.4.1). $\square$

We observe that since

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx,$$

therefore,

$$\begin{aligned}
\widehat{f(t+a)} &= \int_{-\infty}^{\infty} f(x+a) e^{-2\pi i x t} dx \\
&= e^{2\pi i a t} \int_{-\infty}^{\infty} f(y) e^{-2\pi i y t} dy \\
&= e^{2\pi i a t} \widehat{f}(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{f(\frac{t}{a})} &= \int_{-\infty}^{\infty} f\left(\frac{x}{a}\right) e^{-2\pi i x t} dx \\
&= a \int_{-\infty}^{\infty} f(y) e^{-2\pi i y (at)} dy \\
&= a\widehat{f(at)}.
\end{aligned}$$

Thus, we have shown that

Proposition 2.4.3 $\widehat{f(t+a)} = e^{2\pi i a t} \widehat{f}(t)$ *and* $\widehat{f(\frac{t}{a})} = a\widehat{f(at)}$.

It can be shown that ([M2], page 308)

Proposition 2.4.4 *For* $F(x) = e^{-\pi x^2}, \widehat{F}(u) = e^{-\pi u^2}$.

Proof. We have to show that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x u} dx = e^{-\pi u^2}.$$

In other words, we have to show that

$$\int_{-\infty}^{\infty} e^{-\pi(x+iu)^2} dx = 1 \tag{2.1}$$

We will first prove equation (2.1) for $u = 0$ and then show that the value of the integral $\int_{-\infty}^{\infty} e^{-\pi(x+iu)^2} dx$ is independent of u. This will prove equation (2.1).

When $u = 0$, we have the integral

$$I = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

We note that

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-\pi x^2} dx \int_{-\infty}^{\infty} e^{-\pi y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx dy \\ &= \int_{0}^{\infty} r e^{-\pi r^2} dr \int_{0}^{2\pi} d\theta, \end{aligned}$$

by making the polar substitution $x = r\cos\theta$, $y = r\sin\theta$.

Since

$$I^2 = \int_{0}^{\infty} 2\pi r e^{-\pi r^2} dr = 1,$$

and $I > 0$, we deduce $I = 1$.

Also note that

$$\begin{aligned} \frac{\partial}{\partial u} \int_{-\infty}^{\infty} e^{-\pi(x+iu)^2} dx &= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial u} e^{-\pi(x+iu)^2} \right) dx \\ &= i \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} e^{-\pi(x+iu)^2} \right) dx \\ &= \left[i e^{-\pi(x+iu)^2} \right]_{x=-\infty}^{x=\infty} \\ &= 0. \end{aligned}$$

Thus, the value of the integral is independent of u. Since $I = 1$, equation (2.1) is proved.

□

Thus, from Propositions 2.4.3 and 2.4.4, we get

Proposition 2.4.5 *For* $F(x) = e^{-\pi \frac{t^2}{x}}, \widehat{F}(x) = x^{1/2} e^{-\pi t^2 x}$.

Thus, by Poisson summation formula,

Corollary 2.4.6 $\sum_{n\in\mathbb{Z}} e^{-n^2 \frac{\pi}{x}} = x^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-n^2 \pi x}$.

2.5 Properties of the Gamma function

While proving the functional equation of the Riemann zeta function, we will require some properties of the Gamma function $\Gamma(s)$, as outlined in [Ap].

For $Re(s) > 0$, we have the following integral representation:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx.$$

$\Gamma(s)$ can be continued beyond the line $\sigma = 0$, and is analytic everywhere in the s-plane except for simple poles at the points $s = 0, -1, -2, \ldots$ with residue $\frac{(-1)^n}{n!}$ at $s = -n$.

$\Gamma(s)$ also has the representation

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)\cdot \cdots \cdot (s+n)}.$$

for $s \neq 0, -1, -2, \ldots$

The Gamma function satisfies two functional equations

$$\Gamma(s+1) = s\Gamma(s) \tag{2.2}$$

and

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{2.3}$$

It also satisfies a multiplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{m}\right)\ldots\Gamma\left(s+\frac{m-1}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-ms}\Gamma(ms) \tag{2.4}$$

for all s and integers $m \geq 1$.

2.6 Bernoulli numbers and Bernoulli polynomials

Summation of the first n natural numbers, or squares, or cubes, is an elementary problem in number theory. It leads to the following well known formulae,

$$S_1(n) = \sum_{i=0}^{n-1} i = \frac{(n-1)n}{2},$$

$$S_2(n) = \sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6},$$

and

$$S_3(n) = \sum_{i=0}^{n-1} i^3 = \frac{n^2(n-1)^2}{4}.$$

These formulae go back to Aryabhata in the 5th century A.D. who derived the above formulae for $k = 1, 2, 3$ in his book *Aryabhatiya*. (see [**?**], page 38)

The generalization of above formulae to find the sum

$$S_k(n) = \sum_{i=0}^{n-1} i^k$$

is connected with the work *Ars Conjectandi* of Jacob Bernoulli. We define the Bernoulli numbers B_n by the series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

We consider the power series:

$$\begin{aligned}
\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\sum_{j=0}^{n-1} j^k \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{\infty} \frac{(tj)^k}{k!} \\
&= \sum_{j=0}^{n-1} e^{tj} \\
&= \frac{e^{nt}-1}{t}\frac{t}{e^t-1} \\
&= \sum_{i=1}^{\infty} \frac{n^i t^{i-1}}{i!}\sum_{j=0}^{\infty} B_j\frac{t^j}{j!} \\
&= \sum_{i=0}^{\infty} \frac{n^{i+1} t^{i}}{(i+1)!}\sum_{j=0}^{\infty} B_j\frac{t^j}{j!}
\end{aligned}$$

Comparing coefficients of $\frac{t^k}{k!}$, we get

$$S_k(n) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i}\frac{n^{i+1}}{i+1}.$$

Thus $S_k(n)$ is a polynomial in n of degree $k+1$ involving the Bernoulli numbers as its coefficients.

We also observe that

$$\frac{t}{2} + \frac{t}{e^t-1}$$

is an even function of t. Thus, we deduce that $B_k = 0$ for k odd, $k \geq 3$.

We also define the n-th Bernoulli polynomial as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Note that

$$\begin{aligned} S_k(n) &= \sum_{i=0}^{k} \binom{k}{i} B_{k-i} \frac{n^{i+1}}{i+1} \\ &= \frac{1}{k+1} \sum_{i=0}^{k} \frac{k+1}{i+1} \binom{k}{i} B_{k-i} n^{i+1} \\ &= \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i+1} B_{k-i} n^{i+1} \\ &= \frac{1}{k+1} \sum_{i=1}^{k+1} \binom{k+1}{i} B_{k+1-i} n^{i} \\ &= \frac{1}{k+1} \left(B_{k+1}(n) - B_{k+1} \right). \end{aligned}$$

Bernoulli polynomials will appear later as special values of Hurwitz zeta functions.(see Chapter 4)

2.7 Characters mod k and L-Series

Consider the group $(\mathbb{Z}/k\mathbb{Z})^*$ of coprime residue classes mod k.

A homomorphism

$$\chi : (\mathbb{Z}/k\mathbb{Z})^* \longrightarrow (\mathbb{C})^*$$

into the multiplicative group of complex numbers is called a character mod k. We extend the definition of χ to all natural numbers by setting

$$\chi(n) = \begin{cases} \chi(n(mod\, k)) & \text{if } (n,k) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

χ, as defined above, is a completely multiplicative function.

The character

$$\chi_0 : (\mathbb{Z}/k\mathbb{Z})^* \longrightarrow (\mathbb{C})^*$$

satisfying $\chi_0(a) = 1$ for all $(a,k) = 1$ is called the principal character.

The set of characters $(mod\, k)$ forms a group of order $\varphi(k)$, where φ is the Euler $\varphi-$ function. This has been explained in [M2].

We observe that

$$\sum_{r(mod\,k)} \chi_0(r) = \varphi(k).$$

Lemma 2.7.1 *If* $\chi \neq \chi_0$, *then* $\sum_{r(mod\,k)} \chi(r) = 0$.

Proof. Since $\chi \neq \chi_0$, there is a $b(mod\,k)$ such that $(b,k) = 1$ and $\chi(b) \neq 1$.

Then br runs through coprime residue classes $mod\,k$ as r does. Thus,

$$\begin{aligned} s: &= \sum_{r(mod\,k)} \chi(r) \\ &= \sum_{r(mod\,k)} \chi(br) \\ &= \chi(b)s. \end{aligned}$$

Thus,

$$s(\chi(b) - 1) = 0.$$

Since, $\chi(b) \neq 1$, therefore $s = 0$. □

Lemma 2.7.2

$$\sum_{\chi(mod\,k)} \chi(n) = \begin{cases} \varphi(k) & \text{if } n \equiv 1(mod\,k), \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Proof. If $n \equiv 1(mod\,k)$, then

$$\begin{aligned} \sum_{\chi(mod\,k)} \chi(n) &= \sum_{\chi(mod\,k)} 1 \\ &= \varphi(k). \end{aligned}$$

Otherwise, suppose $n \not\equiv 1(mod\,k)$ and $(n,k) = 1$.

Then, there is a character α such that $\alpha(n) \neq 1$. (Refer to page 224 of [M2])

Note that $\alpha\chi$ ranges over all the characters $mod\,k$ as χ does. Thus,

$$\begin{aligned} S: &= \sum_{\chi(mod\,k)} \chi(n) \\ &= \sum_{\chi(mod\,k)} \alpha\chi(n) \\ &= \alpha(n)S. \end{aligned}$$

Since $\alpha(n) \neq 1$, therefore $S = 0$. $\square$

We now define the L-Series $L(s, \chi)$ for $Re(s) > 1$ as follows.

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Notice that when χ is not principal,

$$\sum_{n \leq x} \chi(n) = O(1).$$

So, by corollary 2.3.2, we obtain an analytic continuation of $L(s, \chi)$ for $Re(s) > 0$.

Chapter 3

The Riemann-zeta function and its generalisations

3.1 Analytic continuation of $\zeta(s)$ via partial summation

We recall that the Riemann ζ function was originally defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1.$$

In this section, we will prove the following theorem, using partial summation. This method has also been explained in [MR] and [Ap].

Theorem 3.1.1 $\zeta(s)$ *extends to an analytic function for* $s \in \mathbb{C}$, *apart from* $s = 1$, *where it has a simple pole.*

Proof. From corollary 2.3.3, by partial summation, we have

$$\begin{aligned} \zeta(s) &= s\int_1^{\infty} \frac{[x]}{x^{s+1}} dx \\ &= 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx, \end{aligned}$$

where $\{x\}$ denotes the fractional part of x. This gives us the analytic continuation of $\zeta(s)$ for $Re(s) > 0$.

Next, we write

$$\begin{aligned}\zeta(s) &= 1+\frac{1}{s-1}-s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx \\ &= 1+\frac{1}{s-1}-s\sum_{n=1}^\infty \int_n^{n+1} \frac{x-n}{x^{s+1}}dx \\ &= 1+\frac{1}{s-1}-s\sum_{n=1}^\infty \int_0^1 \frac{u}{(u+n)^{s+1}}du.\end{aligned}$$

Integrating the last integral by parts, we get

$$\sum_{n=1}^\infty \int_0^1 \frac{u}{(u+n)^{s+1}}du = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{(n+1)^{s+1}} + \frac{s+1}{2}\int_1^\infty \frac{\{x\}^2}{x^{s+2}}dx$$

and the second integral converges for $Re(s) > -1$. Thus

$$\zeta(s) = 1+\frac{1}{s-1}-\frac{s}{2}\left(\zeta(s+1)-1\right)-\frac{s(s+1)}{2}\int_1^\infty \frac{\{x\}^2}{x^{s+2}}dx.$$

In this way, $\zeta(s)$ has been analytically continued in the half-plane

$$Re(s) > -1.$$

From above, we also infer that $\zeta(0) = -\frac{1}{2}$. Integration by parts leads to similar representations in successively larger half-planes, as indicated below.

$$\begin{aligned}\zeta(s) = 1 &+ \frac{1}{s-1} \\ &- \sum_{r=1}^m \frac{s(s+1)\dots(s+r-1)}{(r+1)!}\left(\zeta(s+r)-1\right) \\ &- \frac{s(s+1)\dots(s+m)}{(m+1)!}\sum_{n=1}^\infty \int_0^1 \frac{u^{m+1}}{(u+n)^{s+m+1}}du.\end{aligned} \tag{3.1}$$

and the infinite sum on the right hand side converges for $Re(s) > -m$. □

3.2 A new method of analytic continuation of $\zeta(s)$

The aim of this section is to derive the analytic continuation of the Riemann zeta function by an even simpler observation than the one in the previous section. The idea is to use the Hurwitz zeta function for $x = 1/2$. As a consequence, we give a new inductive proof of the well known formula

$$\zeta(1-k) = -B_k/k,$$

where B_k is the k-th Bernoulli number.

The technique is capable of wider generalization. In the last section of this thesis, we will consider generalizations of the classical Hurwitz zeta function and obtain analytic continuations for them. To derive the analytic continuation of $\zeta(s)$, we will make use of the Hurwitz zeta function.

Now, fix $0 < x < 1$.

Recall that the Hurwitz zeta function $\zeta(s,x)$ is defined by the series

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \text{ which converges absolutely for } Re(s) > 1.$$

We notice that

$$\frac{-1}{x^s} + \zeta(s,x) - \zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{(n+x)^s} - \frac{1}{n^s} \right).$$

The summand on the right hand side can be written as

$$\frac{1}{n^s} \left(\left(1 + \frac{x}{n}\right)^{-s} - 1. \right)$$

Using the binomial theorem, we obtain

Proposition 3.2.1 *If $0 < x < 1$, then*

$$\frac{-1}{x^s} + \zeta(s,x) - \zeta(s) = \sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s+r) x^r$$

Proof. Since $0 < x < 1$ and $n \geq 1$, using binomial theorem, we get

$$\left(1 + \frac{x}{n}\right)^{-s} = \sum_{r=0}^{\infty} \binom{-s}{r} \left(\frac{x}{n}\right)^{r}.$$

Thus, we write

$$-\frac{1}{x^s} + \zeta(s, x) - \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{r=1}^{\infty} \binom{-s}{r} \left(\frac{x}{n}\right)^{r}.$$

We note that by ratio test([Ru], page 66), the series of positive terms

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| \left[\sum_{r=1}^{\infty} \left|\binom{-s}{r}\right| \left|\left(\frac{x}{n}\right)^{r}\right|\right]$$

converges for $Re(s) > 1$. So, we can interchange the order of summation to get

$$\frac{-1}{x^s} + \zeta(s, x) - \zeta(s) = \sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s + r) x^r$$

□

The beauty of this proposition is that the right hand side is analytic for $Re(s) > 0$. This is because for $Re(s) > 0$,

$$\left|\sum_{r=1}^{\infty} \binom{-s}{r} x^r \zeta(s + r)\right| = \left|\sum_{r=1}^{\infty} \binom{-s}{r} x^r \sum_{n=1}^{\infty} \frac{1}{n^{\sigma + r}}\right|, \tag{3.2}$$

where $\sigma = Re(s)$. Notice

$$\sigma + r > 1 \, \forall r \geq 1.$$

Then, we note that

$$\begin{aligned} \left|\sum_{r=1}^{\infty} \binom{-s}{r} x^r \sum_{n=1}^{\infty} \frac{1}{n^{\sigma + r}}\right| &\leq \sum_{r=1}^{\infty} \left|\binom{-s}{r}\right| x^r \sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma}} \\ &= \sum_{r=1}^{\infty} C \left|\binom{-s}{r}\right| x^r, \end{aligned}$$

where

$$C = \sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma}}.$$

Once again, by ratio test,

$$\sum_{r=1}^{\infty} C \left| \binom{-s}{r} \right| x^r$$

is convergent.

Thus, by Weierstrass M-test, ([Ru], page 148),

$$\sum_{r=1}^{\infty} \binom{-s}{r} x^r \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+r}}$$

is uniformly convergent for $Re(s) > 0$.

It follows that

$$\sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s+r) x^r$$

converges uniformly in every compact subset of the half plane

$$\{s \in \mathbb{C} : Re(s) > 0\}.$$

Thus,

$$\sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s+r) x^r$$

is analytic for $Re(s) > 0$.

We will exploit this fact below. However, before we do this, let us note that

$$\begin{aligned}
\zeta(s, 1/2) &= 2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \\
&= 2^s(\zeta(s) - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}) \\
&= 2^s(\zeta(s) - 2^{-s}\zeta(s)) \\
&= (2^s - 1)\zeta(s).
\end{aligned}$$

In this way, we get

Lemma 3.2.2 $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$.

This is the key observation that we will use to analytically continue $\zeta(s)$. In Proposition 3.2.1, for $x = 1/2$, we get

$$-2^s + \zeta(s, 1/2) - \zeta(s) = \sum_{r=1}^{\infty} \binom{-s}{r} (\frac{1}{2})^r \zeta(s+r).$$

Using Lemma 3.2.2, we deduce

Theorem 3.2.3

$$(2^s - 2)\zeta(s) = 2^s + \sum_{r=1}^{\infty} \binom{-s}{r} \left(\frac{1}{2}\right)^r \zeta(s+r).$$

Since the right hand side converges for $Re(s) > 0$, we obtain an analytic continuation of $(2^s - 2)\zeta(s)$ in this region. It is immediate that this gives a meromorphic continuation of $\zeta(s)$ for $Re(s) > 0$ with possible poles only at $s = 1 + \frac{2\pi i m}{\log 2}$, where m is an integer. It is also interesting that we can use Theorem 3.2.3 to derive (by induction) a meromorphic continuation of $\zeta(s)$ to the entire complex plane. We do this below.

Note first that

$$\begin{aligned}
\sum_{r=1}^{\infty} \binom{-1}{r} (\frac{1}{2})^r \zeta(1+r) &= \sum_{r=1}^{\infty} (-1)^r (\frac{1}{2})^r \sum_{n=1}^{\infty} \frac{1}{n^{r+1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{r=1}^{\infty} (\frac{-1}{2n})^r \\
&= -2 \sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{s \to 1} (2^s - 2)\zeta(s) &= 2 - 2\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)} \\
&= 2 - 2\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1}\right) \\
&= 2\log 2.
\end{aligned}$$

So we obtain, as a corollary to Theorem 3.2.3, the following

Corollary 3.2.4 $(2^s - 2)\zeta(s)$ *extends to an analytic function for* $Re(s) > 0$ *and*

$$\lim_{s\to 1}(2^s - 2)\zeta(s) = 2\log 2.$$

Thus

$$\lim_{s\to 1}\frac{2^s - 2}{s-1}(s-1)\zeta(s) = 2\log 2$$

so that

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

There is nothing special about $\zeta(s, \frac{1}{2})$. Lemma 3.2.2 can be generalized as follows :

Lemma 3.2.5

$$(m^s - m)\zeta(s) = \sum_{a=1}^{m}\left[\zeta(s, \frac{a}{m}) - \zeta(s)\right].$$

Proof.

$$m^{-s}\sum_{a=1}^{m}\zeta(s, \frac{a}{m}) = \sum_{a=1}^{m}\sum_{n=0}^{\infty}\frac{1}{(mn+a)^s} = \zeta(s)$$

Thus,

$$m^s\zeta(s) = \sum_{a=1}^{m}\zeta(s, \frac{a}{m})$$

Thus,

$$(m^s - m)\zeta(s) = \sum_{a=1}^{m}\left[\zeta(s, \frac{a}{m}) - \zeta(s)\right].$$

□

The interesting feature of Lemma 3.2.5 is that the right hand side is analytic for $Re(s) > 0$ by Proposition 3.2.1 and the remark afterwards. Hence, by the same logic as used in the remarks following Theorem 3.2.3, we deduce that $\zeta(s)$ extends to a meromorphic function for $Re(s) > 0$ with possible poles at $s = 1 + \frac{2\pi i k}{\log m}, k \in \mathbb{Z}$. Choosing $m = 3$, we find that the only possible poles of $\zeta(s)$ lie in

$$\left\{1 + \frac{2\pi i k}{\log 3} : k \in \mathbb{Z}\right\} \cap \left\{1 + \frac{2\pi i l}{\log 2} : l \in \mathbb{Z}\right\} = \{1\}.$$

This shows that $\zeta(s)$ extends to an analytic function for $Re(s) > 0$ except for a simple pole at $s = 1$. From proposition 3.2.1 and Lemma 3.2.2, we have

$$(2^s - 2)\zeta(s) = 2^s + \sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s+r)2^{-r}. \tag{3.3}$$

Let us now assume that $\zeta(s)$ is analytic for $Re(s) > -(m-1)$, $m \in N$.

Equation (3.3) can be rewritten as

$$(2^s - 2)\zeta(s) = 2^s + \sum_{r=1}^{m} \binom{-s}{r} 2^{-r}\zeta(s+r) + \sum_{r=m+1}^{\infty} \binom{-s}{r} 2^{-r}\zeta(s+r).$$

If $Re(s) > -m$, then

$$\sum_{r=1}^{m} \binom{-s}{r} 2^{-r}\zeta(s+r),$$

being a finite sum of analytic functions is analytic.

From arguments following Proposition 3.2.1, we deduce that

$$\sum_{r=m+1}^{\infty} \binom{-s}{r} 2^{-r}\zeta(s+r)$$

is analytic for $Re(s) > -m$.

So, assuming that $\zeta(s)$ has an analytic continuation in the half plane

$$Re(s) > -(m-1),\ m \in \mathbb{N}$$

we have shown that $\zeta(s)$ can be analytically continued in the region

$$Re(s) > -m.$$

Thus, by induction, we see that $\zeta(s)$ extends analytically for all $s \in \mathbb{C}$, except for a simple pole of residue 1 at $s = 1$.

This gives us an elementary proof of Theorem 3.1.1.

3.3 A new derivation of $\zeta(1-k) = -\frac{B_k}{k}$

We now use the derivation in the previous section to obtain a new proof of the formula

$$\zeta(1-k) = -\frac{B_k}{k},\ k \geq 2.$$

Since

$$\frac{t}{\exp(t)-1} + \frac{t}{2}$$

is an even function, the Bernoulli numbers for odd subscripts ≥ 3 vanish, and we can write the above formula as

Theorem 3.3.1

$$\zeta(1-k) = (-1)^{k-1} B_k/k, \; k \geq 1.$$

Proof.

Observe that equation (3.3) can be written as

$$(2^s - 2)\zeta(s) = 2^s - s\frac{\zeta(s+1)}{2} + \sum_{r=2}^{\infty} \binom{-s}{r} \zeta(s+r) 2^{-r}$$

where we see that the right hand side is analytic for $Re(s) > -1$. We may substitute $s = 0$ in the above formula and deduce that

$$\zeta(0) = -\frac{1}{2}.$$

Since $B_1 = -\frac{1}{2}$, the above theorem holds for $k = 1$.

We proceed by induction on k.

Suppose the result holds for all $m \in \mathbb{N}$, $m < k$.

We note that for $r = k$, $\zeta(1-k+r)$ has a simple pole of residue 1. So,

$$\lim_{r \to k} (r-k)\zeta(1-k+r) = 1.$$

Then,

$$\lim_{r \to k} \binom{k-1}{r} \frac{1}{2^r} \zeta(1-k+r) = -\frac{1}{2^k k}.$$

Also, if $r > k$, the term

$$\binom{k-1}{r} \frac{1}{2^r} \zeta(1-k+r)$$

vanishes, since $\zeta(1-k+r)$ is analytic. Thus, if we put $s = 1-k$ in Theorem (3.2.3), we obtain the recurrence

$$(2^{1-k} - 2)\zeta(1-k) = 2^{1-k} + \sum_{r=1}^{k-1} \binom{k-1}{r} \frac{1}{2^r} \zeta(1-k+r) - \frac{1}{2^k k}. \tag{3.4}$$

Then, by induction hypothesis and the fact that $\zeta(0) = B_1$, we get

$$\zeta(1-k+r) = (-1)^{k-r-1}\frac{B_{k-r}}{k-r},\ 1 \leq r \leq (k-1).$$

So, the right hand side of equation (3.5) becomes

$$\begin{aligned} & 2^{1-k} + \sum_{r=1}^{k-1} \binom{k-1}{r} \frac{1}{2^r}(-1)^{k-r-1}\frac{B_{k-r}}{k-r} - \frac{1}{2^k k} \\ &= 2^{1-k} + \frac{1}{k}\sum_{r=1}^{k-1} \binom{k}{r} \frac{1}{2^r}(-1)^{k-r-1}B_{k-r} - \frac{1}{2^k k} \qquad (3.5) \\ &= 2^{1-k} + \frac{1}{k}\sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r}(-1)^{k-r-1}B_{k-r} - \frac{1}{k}(-1)^{k-1}B_k. \end{aligned}$$

Thus, we get

$$(2^{1-k}-2)\zeta(1-k) = 2^{1-k} - (-1)^{k-1}\frac{B_k}{k} - \frac{1}{k}\sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r}(-1)^{k-r}B_{k-r}.$$

Let us define $c_k s$ as follows.

$$\sum_{k=0}^{\infty} c_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{2^k k!} \sum_{k=0}^{\infty} (-1)^k B_k \frac{x^k}{k!}. \qquad (3.6)$$

Then, $c_k = \sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r}(-1)^{k-r}B_{k-r}$ for every $k \geq 1$.
Note that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{c_k x^k}{k!} &= \exp(\frac{x}{2})\frac{(-x)}{\exp(-x)-1} \\ &= x\exp\left(\frac{x}{2}\right) + \frac{x}{\exp(\frac{x}{2}) - \exp\left(\frac{-x}{2}\right)}. \end{aligned}$$

Now, coefficient of $\frac{x^k}{k!}$ in $x\exp(\frac{x}{2})$ is $\frac{k}{2^{k-1}}$.
We have to determine the coefficient of $\frac{x^k}{k!}$ in

$$x\exp\left(\frac{x}{2}\right) + \frac{x}{\exp(\frac{x}{2}) - \exp\left(\frac{-x}{2}\right)} = x\exp\left(\frac{x}{2}\right) + \frac{ix}{2}\left(\sin(\frac{ix}{2})\right)^{-1}$$

Recall from [Co](page 88) that

$$(\sin(x))^{-1}=x^{-1}+\sum_{m\geq 1}B_{2m}(-1)^{m+1}(2^{2m}-2)\frac{x^{2m-1}}{(2m)!}$$

Using this formula, we can deduce that

$$\frac{ix}{2}\left(\sin(\frac{ix}{2})\right)^{-1}=1-\sum_{k=1}^{\infty}B_{2k}(2^{2k}-2)\left(\frac{x}{2}\right)^{2k}\frac{1}{(2k)!}$$

Thus, **if k is even** ,

$$c_k=\frac{k}{2^{k-1}}-B_k\frac{2^k-2}{2^k}$$

As a result,

$$\begin{aligned}(2^{1-k}-2)\zeta(1-k) & \\ &= 2^{1-k}-(-1)^{k-1}B_k-\frac{1}{k}\left\{\frac{k}{2^{k-1}}-B_k\frac{(2^k-2)}{2^k}\right\}\\ &= (2-2^{1-k})\frac{B_k}{k}\end{aligned}$$

that is, $\zeta(1-k)=-\frac{B_k}{k}$

If **k is odd** ,

$$c_k=\frac{k}{2^{k-1}}.$$

Thus

$$(2^{1-k}-2)\zeta(1-k)=-(-1)^{k-1}\frac{B_k}{k}.$$

In particular, if $k=1$, then $\zeta(0)=-\frac{1}{2}=B_1$

If $k\geq 3$, then $\zeta(1-k)=0=B_k$ for k odd.

The theorem for $m=k$ follows from the different cases considered above .

Thus, we have proved our result by induction . □

Corollary 3.3.2 *$\zeta(s)=0$ for $s=-2n, n\in\mathbb{N}$.*

Proof. We know that $B_k = 0$ for odd $k \in \mathbb{Z}, k \geq 3$. Thus,

$$\begin{aligned}\zeta(-2n) &= -\frac{B_{2n+1}}{2n+1} \\ &= 0.\end{aligned}$$

□

3.4 An identity involving $\zeta(2n)$

This section is based on [GW]. In this paper, G.T.Williams presented, among other things, a new method of evaluating $\zeta(2n)$. His method is elementary and involves manipulations of the defining series.

He proves the following

Theorem 3.4.1

$$\zeta(2)\zeta(2n-2) + \zeta(4)\zeta(2n-4) + \ldots \zeta(2n-2)\zeta(2) = (n + \frac{1}{2})\zeta(2n).$$

This convolution serves to give $\zeta(2n)$, recursively, as a rational multiple of $(\zeta(2))^n$.

Proof. The left hand side is the limit, as $N \to \infty$, of

$$\sum_{\nu=1}^{N}\sum_{\mu=1}^{N}\{\frac{1}{\mu^2}\frac{1}{\nu^{2n-2}} + \frac{1}{\mu^4}\frac{1}{\nu^{2n-4}} + \cdots + \frac{1}{\mu^{2n-2}}\frac{1}{\nu^2}\} \tag{3.7}$$

Here, N is a positive integer. Summing the geometric series within braces, and taking note of the exceptional case when $\mu = \nu$, this becomes

$$\sum_{\nu}\sum_{\mu}{}' \frac{\nu^{2-2n} - \mu^{2-2n}}{\mu^2 - \nu^2} + (n-1)\sum_{\nu}\frac{1}{\nu^{2n}}. \tag{3.8}$$

Throughout this proof, all sums run from 1 to N and an accent on the inner sum indicates that the index(in this case, μ) does not take on the value of the index of the outer sum (ν here.)

Ignoring the term on the far right, (3.4) is equal to

$$\sum_{\nu}\sum_{\mu}{}' \frac{\nu^{2-2n}}{\mu^2 - \nu^2} + \sum_{\nu}\sum_{\mu}{}' \frac{\mu^{2-2n}}{\nu^2 - \mu^2}. \tag{3.9}$$

Inverting the order of summation in the second double sum, and noting that the condition $\mu \neq \nu$ is the same as the condition $\nu \neq \mu$, (3.5) can be written as

$$2\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2} \tag{3.10}$$

Combining the above equations, we find that equation 3.3 can be written as

$$(n-1)\sum_{\nu}\frac{1}{\nu^{2n}}+2\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2}. \tag{3.11}$$

Now,

$$\begin{aligned} 2\nu\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2} &= \sum_{\mu}{}'\frac{1}{\mu-\nu}-\sum_{\mu}{}'\frac{1}{\mu+\nu} \\ &= -\sum_{\mu=1}^{N+\nu}\frac{1}{\mu}+\frac{1}{\nu}+\sum_{\mu=1}^{N-\nu}\frac{1}{\mu}+\frac{1}{2\nu} \\ &= \frac{3}{2\nu}-\left\{\frac{1}{N-\nu+1}+\cdots+\frac{1}{N+\nu}\right\}. \end{aligned}$$

When we substitute this into (3.7), we get that (3.3) is equal to

$$(n+\frac{1}{2})\sum_{\nu}\frac{1}{\nu^{2n}}-\sum_{\nu}\frac{1}{\nu^{2n-1}}\left(\frac{1}{N-\nu+1}+\cdots+\frac{1}{N+\nu}\right) \tag{3.12}$$

Finally,

$$\frac{1}{N-\nu+1}+\frac{1}{N-\nu+2}+\cdots+\frac{1}{N+\nu} \quad < \quad \frac{2\nu}{N-\nu+1}$$

and so

$$\begin{aligned} 0 &< \sum_{\nu} \frac{1}{\nu^{2n-1}} \left(\frac{1}{N-\nu+1} + \cdots + \frac{1}{N+\nu} \right) \\ &< 2\sum_{\nu} \frac{1}{\nu^{2n-2}} . \frac{1}{N-\nu+1} \\ &\leq 2\sum_{\nu} \frac{1}{\nu(N-\nu+1)} \\ &= \frac{2}{N+1} \sum_{\nu} \left(\frac{1}{\nu} + \frac{1}{N-\nu+1} \right) \\ &= \frac{4}{N+1} \sum_{\nu=1}^{N} \frac{1}{\nu} \\ &< \frac{4}{N+1} (1+\log(N)) \end{aligned}$$

We observe that

$$\frac{4}{N+1}(1+\log(N)) \to 0$$

as $N \to \infty$. The above statement and equation (3.8) taken together give us the theorem.

□

3.5 Functional equation of the Riemann zeta function

We now derive the functional equation of the Riemann Zeta function.

$\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)$$

The functional equation can be written in a number of ways. Changing s into $1-s$ and using the functional equation for $\Gamma(s)$, it becomes

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s).$$

It can also be written in the symmetric form

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)$$

and we will prove the functional equation in the above form.

We define the Θ function as

$$\Theta(z) = \sum_{n\in\mathbb{Z}} e^{-\pi i n^2 z}, z \in \mathbb{C}, Im(z) > 0.$$

Now, define $\omega(y) = \Theta(iy)$. Thus,

$$\omega(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$$

and

$$\omega\left(\frac{1}{x}\right) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2/x}.$$

We know, from corollary (2.4.6), that

$$\omega\left(\frac{1}{x}\right) = x^{1/2}\omega(x).$$

Now,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}t^{s/2}\frac{dt}{t}.$$

Put $t = n^2\pi x$. Then,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-n^2\pi x}n^s\pi^{s/2}x^{s/2}\frac{dx}{x}.$$

$$\Rightarrow \pi^{-s/2}n^{-s}\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-n^2\pi x}x^{s/2}\frac{dx}{x}$$

For $Re(s) > 1$,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \left(\sum_{n=1}^\infty e^{-n^2\pi x}\right)x^{s/2}\frac{dx}{x}.$$

Note that $1+2\sum_{n=1}^{\infty} e^{-n^2\pi x} = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$. So,

$$\sum_{n=1}^{\infty} e^{-n^2\pi x} = \frac{\omega(x)-1}{2}.$$

Let us set $W(x) = \frac{\omega(x)-1}{2}$.

Now,

$$\begin{aligned}
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^{\infty} x^{s/2}\left(\frac{\omega(x)-1}{2}\right)\frac{dx}{x} \\
&= \int_0^{\infty} x^{s/2}W(x)\frac{dx}{x} \\
&= \int_1^{\infty} x^{s/2}W(x)\frac{dx}{x} + \int_0^{1} x^{s/2}W(x)\frac{dx}{x} \\
&= \int_1^{\infty} x^{s/2}W(x)\frac{dx}{x} + \int_1^{\infty} x^{-s/2}W\left(\frac{1}{x}\right)\frac{dx}{x}.
\end{aligned}$$

Now,

$$\begin{aligned}
W\left(\frac{1}{x}\right) &= \frac{\omega\left(\frac{1}{x}\right)-1}{2} \\
&= \frac{x^{1/2}\omega(x)-1}{2} \\
&= \frac{x^{1/2}\{2W(x)+1\}-1}{2} \\
&= -\frac{1}{2}+\frac{1}{2}x^{1/2}+x^{1/2}W(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_1^{\infty} W\left(\frac{1}{x}\right)x^{-s/2}\frac{dx}{x} \\
&\qquad = \int_1^{\infty}\left(-\frac{1}{2}+\frac{1}{2}x^{1/2}+x^{1/2}W(x)\right)x^{-s/2}\frac{dx}{x} \\
&\qquad = -\frac{1}{s}+\frac{1}{s-1}+\int_1^{\infty} x^{(1-s)/2}W(x)\frac{dx}{x}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) & \\
&= \int_1^\infty x^{s/2}W(x)\frac{dx}{x} + \int_1^\infty x^{(1-s)/2}W(x)\frac{dx}{x} + \frac{1}{s-1} - \frac{1}{s} \\
&= \int_1^\infty \left(x^{s/2} + x^{(1-s)/2}\right) W(x)\frac{dx}{x} + \frac{1}{s-1} - \frac{1}{s}.
\end{aligned}$$

Thus, it follows from above that

$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)$$

This is Riemann's derivation of the analytic continuation of $\zeta(s)$ and functional equation.

Chapter 4

The Hurwitz zeta function

4.1 Classical Hurwitz zeta function

In the last chapter, we defined the Hurwitz zeta function :

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where we fix $0 < a \leq 1$. The above series converges absolutely for $Re(s) > 1$.

For $Re(s) > 1$, partial summation gives us

$$\zeta(s,a) = s\int_0^{\infty} \frac{[x]}{(x+a)^{s+1}} dx.$$

As in section 3.1, integration by parts leads to the formula

$$\begin{aligned} \zeta(s,a) = \frac{a^{1-s}}{s-1} - \sum_{r=1}^{m} \frac{s(s+1)\dots(s+r-1)}{(r+1)!} \left(\zeta(s+r,a) - a^{-s-r}\right) \\ + \frac{s(s+1)\dots(s+m)}{(m+1)!} \sum_{n=1}^{\infty} \int_0^1 \frac{u^{m+1}}{(u+n+a)^{s+m+1}} du \end{aligned} \tag{4.1}$$

The infinite sum on right hand side converges for $Re(s) > -m$. Thus, the method used for Riemann zeta function generalizes to give the analytic continuation of the Hurwitz zeta function $\zeta(s,a)$ for real values of a between 0 and 1.

However, we can consider the Hurwitz zeta function for complex values of a, provided we are careful that $\log(a)$ is well-defined. We do this below and derive an analytic continuation of $\zeta(s,a)$. As before, we can inquire into the nature of the special values $\zeta(1-k,a)$. This turns out to be $-B_k(a)/k$ for appropriate complex values of a.

4.2 Hurwitz zeta function $\zeta(s,a)$ for complex values of a

Our new method to analytically continue $\zeta(s)$, explained in section 3.2, is a versatile method because it enables us to analytically continue $\zeta(s,a)$, where a belongs a much larger complex domain $\mathbb{D}$, where

$$\mathbb{D} = \{z : |z| < 1\} \setminus \{z \in \mathbb{R} : -1 < z \leq 0\}$$

We consider the series

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$$

where z is a complex number. The summand is to be interpreted as

$$\exp(-s \log(n+z))$$

and therefore, for the logarithm to be defined, we need to have z not lie on the negative real axis. We note that

$$\frac{-1}{z^s} + \zeta(s,z) - \zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{(n+z)^s} - \frac{1}{n^s} \right).$$

Writing the summand as

$$\frac{1}{n^s}\left(\left(1+\frac{z}{n}\right)^{-s} - 1\right)$$

and using the binomial theorem, we obtain

Theorem 4.2.1 *For $z \in \mathbb{D}$, we have*

$$\frac{-1}{z^s} + \zeta(s,z) - \zeta(s) = \sum_{r=1}^{\infty} \binom{-s}{r} \zeta(s+r) z^r.$$

Since the right hand side converges for $Re(s) > 0$, we obtain an analytic continuation of $\zeta(s,z)$ in this region. Analogous to remarks made after theorem 3.2.3, we can derive, by induction the following result.

Theorem 4.2.2 *The function $\zeta(s,z)$ for $z \in \mathbb{D}$ is analytic in the whole complex plane except for a simple pole at $s = 1$.*

Furthermore, given any non-zero complex number z not lying in the negative real axis, we can find a positive integer m such that $|z| < m$. Now,

$$-\frac{1}{z^s} + \zeta(s,z) - \zeta(s) = \sum_{n=1}^{m-1} \frac{1}{n^s}\left(\left(1+\frac{z}{n}\right)^{-s} - 1\right) + \sum_{n=m}^{\infty} \frac{1}{n^s}\left(\left(1+\frac{z}{n}\right)^{-s} - 1\right)$$

Since, $|z| < m$, therefore $\left|\frac{z}{n}\right| < 1 \, \forall n \geq m$.

Thus, applying binomial theorem to the second term on the right hand side and by a change in the order of summation, we obtain a modification of Theorem 4.2.2.

Theorem 4.2.3 *Let*

$$\mathbb{D}' = \{z \in \mathbb{C}\} \setminus \{z \in \mathbb{R} : \, z \leq 0\}.$$

Then, given $z \in \mathbb{D}'$, $\exists m \in \mathbb{N}$ such that

$$-\frac{1}{z^s} + \zeta(s,z) - \zeta(s) = \sum_{n=1}^{m-1} \frac{1}{n^s}\left(\left(1+\frac{z}{n}\right)^{-s} - 1\right) + \sum_{r=1}^{\infty} \binom{-s}{r} z^r \left(\zeta(s+r) - \sum_{n=1}^{m-1} \frac{1}{n^{s+r}}\right).$$

Since the right hand side is analytic for $Re(s) > 0$, we obtain an analytic continuation of $\zeta(s,z)$ in this region. Thus, by induction, we can derive that the function $\zeta(s,z)$, for $z \in \mathbb{D}'$ is analytic in the whole complex plane except for a simple pole at $s = 1$.

4.3 Derivation of $\zeta(1-k,z) = -\frac{B_k(z)}{k}$

We now prove the following result:

Theorem 4.3.1 *For $z \in \mathbb{D}'$,*

$$\zeta(1-k,z) = -\frac{B_k(z)}{k}, \; k \in \mathbb{Z}, \; k \geq 2.$$

Proof. We first prove the theorem for $z \in \mathbb{D}$. From Theorem 4.2.1, we have

$$\frac{-1}{z^s} + \zeta(s,z) - \zeta(s) = \sum_{r=1}^{\infty} (-1)^r \frac{(s)(s+1)\dots(s+r-1)}{r!} \zeta(s+r) z^r \quad (4.2)$$

In equation (4.2), put $s = 1-k$ and observe that if $r > k$, then the sum vanishes, since $\zeta(1-k+r)$ is analytic.

At $r = k, \zeta(1-k+r)$ has a simple pole of residue 1. Thus,

$$\lim_{r \to k} (r-k)\zeta(1-k+r) = 1.$$

This implies that

$$\begin{aligned}
\lim_{r \to k} (-1)^r \frac{(1-k)(2-k)\dots(r-1-k)(r-k)}{r!} \zeta(1-k+r) &= \frac{(-1)^k}{k!}(-1)^{k-1}(k-1)! \\
&= -\frac{1}{k}.
\end{aligned}$$

So,

$$-\frac{1}{z^{1-k}} + \zeta(1-k,z) - \zeta(1-k) = \sum_{r=1}^{k-1} \binom{k-1}{r} \zeta(1-k+r) z^r - \frac{z^k}{k}. \quad (4.3)$$

Thus,

$$\begin{aligned}
\zeta(1-k,z) &= z^{k-1} + \zeta(1-k) + \sum_{r=1}^{k-1} \binom{k-1}{r} \zeta(1-k+r) z^r - \frac{z^k}{k} \\
&= z^{k-1} + \zeta(1-k) - \sum_{r=1}^{k-2} \binom{k-1}{r} \frac{B_{k-r}}{k-r} z^r + \zeta(0) z^{k-1} - \frac{z^k}{k} \\
&= \frac{z^{k-1}}{2} + \zeta(1-k) - \frac{1}{k} \sum_{r=1}^{k-2} \binom{k}{r} B_{k-r} z^r - \frac{z^k}{k} \\
&= -B_1 z^{k-1} - \frac{B_k}{k} - B_0 \frac{z^k}{k} - \frac{1}{k} \sum_{r=1}^{k-2} \binom{k}{r} B_{k-r} z^r \\
&= -\frac{1}{k} \sum_{r=0}^{k} \binom{k}{r} B_{k-r} z^r \\
&= -\frac{B_k(z)}{k}
\end{aligned}$$

More generally, let $z \in \mathbb{D}'$. From Theorem 4.2.3, we have

$$-\frac{1}{z^s}+\zeta(s,z)-\zeta(s)=\sum_{n=1}^{m-1}\frac{1}{n^s}\left(\left(1+\frac{z}{n}\right)^{-s}-1\right)+\sum_{r=1}^{\infty}\binom{-s}{r}z^r\zeta(s+r) - \sum_{r=1}^{\infty}\binom{-s}{r}z^r\left(\sum_{n=1}^{m-1}\frac{1}{n^{s+r}}\right). \tag{4.4}$$

In the above equation, we put $s=1-k$ where $k\in\mathbb{Z}$, $k\geq 2$.

Then, we get the following equation :

$$-z^{k-1}+\zeta(1-k,z)-\zeta(1-k)=\sum_{n=1}^{m-1}\frac{1}{n^{1-k}}\left(\left(1+\frac{z}{n}\right)^{k-1}-1\right)+\sum_{r=1}^{k-1}\binom{k-1}{r}\zeta(1-k+r)z^r - \frac{z^k}{k}-\sum_{r=1}^{k-1}\binom{k-1}{r}z^r\sum_{n=1}^{m-1}\frac{1}{n^{1-k+r}}. \tag{4.5}$$

Now, the first term on the right hand side in equation 4.5 is

$$\begin{aligned}\sum_{n=1}^{m-1}\frac{1}{n^{1-k}}\left[\sum_{r=0}^{k-1}\binom{k-1}{r}\left(\frac{z}{n}\right)^r-1\right] &= \sum_{n=1}^{m-1}\frac{1}{n^{1-k}}\left[\sum_{r=1}^{k-1}\binom{k-1}{r}\left(\frac{z}{n}\right)^r\right]\\ &= \sum_{r=1}^{k-1}\binom{k-1}{r}z^r\sum_{n=1}^{m-1}\frac{1}{n^{1-k+r}}.\end{aligned}$$

Thus, equation 4.5 reduces to equation 4.3.

Since equation 4.3 holds for all $z\in\mathbb{D}'$, our result is proved.

□

The idea of this proof originates in [MR].

4.4 L-Series and its relation to $\zeta(s, x)$

If χ is a character $mod\, k$, we rearrange the terms in the series for $L(s,\chi)$ according to the residue classes $mod(k)$. That is, we write

$$n = qk + r, 1 \leq r \leq k, \quad q = 0, 1, 2, \ldots$$

and obtain

$$\begin{aligned} L(s,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= \sum_{q=0}^{\infty} \sum_{r=1}^{k} \frac{\chi(qk+r)}{(qk+r)^s} \\ &= \sum_{r=1}^{k} \frac{1}{k^s} \sum_{q=0}^{\infty} \frac{\chi(r)}{\left(q+\frac{r}{k}\right)^s} \\ &= k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, \frac{r}{k}). \end{aligned}$$

Theorem 4.4.1 *For the principal character χ_0, the L-function $L(s,\chi_0)$ is analytic everywhere except for a simple pole at $s = 1$ with residue $\frac{\varphi(k)}{k}$.*

If $\chi \neq \chi_0, L(s,\chi)$ is an entire function of s. Furthermore, if we define the generalized Bernoulli polynomial $B_{n,\chi}$, by

$$B_{n,\chi} = k^{n-1} \sum_{r=1}^{k} \chi(r) B_n(r/k),$$

then,

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}.$$

Proof. From Lemma 2.7.1, we have the the following

$$\sum_{r\, mod\, k} \chi(r) = \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ \varphi(k) & \text{if } \chi = \chi_0. \end{cases} \tag{4.6}$$

Since $\zeta(s, r/k)$ has a simple pole at $s = 1$ with residue 1, the function $\chi(r)\zeta(s, r/k)$ has a simple pole at $s = 1$ with residue $\chi(r)$. Therefore

$$\begin{aligned}
\operatorname*{Res}_{s=1} L(s,\chi) &= \lim_{s\to 1}(s-1)L(s,\chi) \\
&= \lim_{s\to 1}(s-1)k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s,\frac{r}{k}) \\
&= \frac{1}{k}\sum_{r=1}^{k}\chi(r) \\
&= \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ \frac{\varphi(k)}{k} & \text{if } \chi = \chi_0. \end{cases}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&L(1-n,\chi) \\
&\quad = \sum_{r=1}^{k}\chi(r)k^{n-1}\zeta(1-n,\frac{r}{k}) \\
&\quad = -\sum_{r=1}^{k}\chi(r)k^{n-1}\frac{B_n(r/k)}{n} \\
&\quad = -\frac{B_{n,\chi}}{n}.
\end{aligned}$$

This proves our theorem.

□

Chapter 5

Generalised Hurwitz zeta series

5.1 Generalization of the Hurwitz Zeta Function

In this chapter, we make a few final remarks about how the definition of $\zeta(s,z)$ for complex z can be extended more generally to define what we call the Generalized Hurwitz zeta function. This function has applications in a more general theory of regularized products of certain sequences satisfying a few basic axioms[JL]. A basic example has been presented below.

Let Λ be a sequence $\{z_n\}, z_n \in \mathbb{D}$.

We define the Generalized Hurwitz Zeta function

$$\zeta(s,\Lambda) = \sum_{n=0}^{\infty}(n+z_n)^{-s}.$$

This series is absolutely convergent for $Re(s) > 1$. As a further generalization of Proposition 3.2.1, we obtain

Proposition 5.1.1

$$\zeta(s,\Lambda) = \frac{1}{(z_0)^s} + \zeta(s) + \sum_{r=1}^{\infty}\binom{-s}{r}\sum_{n=1}^{\infty}\frac{z_n^r}{n^{s+r}}.$$

Define

$$L_r(s) = \sum_{n=1}^{\infty}\frac{z_n^r}{n^{s+r}}$$

where $s \in \mathbb{C}$, $r \in \mathbb{N}$. The above series converges absolutely for $Re(s) > 1 - r$. It immediately follows that $\zeta(s, \Lambda)$ is analytic for $Re(s) > 0$ except at $s = 1$ where it has a simple pole.

Then, we consider the derivative

$$\zeta'(s, \Lambda) = -\sum_{n=1}^{\infty} \frac{\log(z_n + n)}{(z_n + n)^s}$$

Putting $s = 0$ formally, we find

$$\zeta'(0, \Lambda) \sim -\sum_{n=1}^{\infty} \log(z_n + n)$$

If we assume that $L_1(s)$ analytically extends to $s = 0$, then $\zeta(s, \Lambda)$ has an analytic continuation at $s = 0$.

Therefore,

$$\exp(-\zeta'(0, \Lambda)) \sim \prod_{n=1}^{\infty} (n + z_n)$$

We thus obtain an interpretation of the infinite product $\{n + z_n\}$, which is called a regularized product. The notion is useful in certain cases to give meaning to the determinant of a linear operator T with eigenvalues given by a sequence $\{\lambda_n\}_{n=1}^{\infty}$. Thus, we may define

$$det\, T = \prod_{n=1}^{\infty} \lambda_n$$

where the infinite product is defined via the formalism above. Namely, let

$$Z(s) = \sum_{n=1}^{\infty} (\lambda_n)^{-s}.$$

Suppose $Z(s)$ admits an analytic continuation for $Re(s) \geq 0$.

Then, as

$$-Z'(s) = \sum_{n=1}^{\infty} (\log \lambda_n) \lambda_n^{-s}$$

we formally have

$$-Z'(0) = \sum_{n=1}^{\infty} \log \lambda_n.$$

Hence, we may interpret $exp(-Z'(0))$ as the determinant of the operator T.

5.2 Poly-Hurwitz zeta functions

We define poly-Hurwitz zeta functions as follows

$$\zeta(s_1, s_2, \ldots, s_r; x_1, x_2, \ldots, x_r) = \sum_{n=0}^{\infty}(n + x_1)^{-s_1} \ldots (n + x_r)^{-s_r},$$

for $0 < x_i \leq 1$. This series absolutely converges when $Re(s_1+s_2+\cdots+s_r) > 1$. Now, let

$$\zeta^*(s_1, s_2, \ldots, s_r; x_1, x_2, \ldots, x_r) = \sum_{n=1}^{\infty}(n + x_1)^{-s_1} \ldots (n + x_r)^{-s_r},$$

Then,

$$\begin{aligned} \zeta^*(s_1, s_2, \ldots, s_r; x_1, x_2, \ldots, x_r) &= \sum_{n=1}^{\infty} \frac{1}{n^{s_1}}.\frac{1}{n^{s_2}} \ldots \frac{1}{n^{s_r}} \left(1 + \frac{x_1}{n}\right)^{-s_1} \ldots \left(1 + \frac{x_r}{n}\right)^{-s_r} \\ &= \sum_{j_1,\ldots,j_r} \binom{-s_1}{j_1}\binom{-s_2}{j_2} \ldots \binom{-s_r}{j_r} x_1^{j_1} \ldots .x_r^{j_r} A, \end{aligned}$$

where $A = \zeta(s_1 + s_2 + \cdots + s_r + j_1 + \cdots + j_r)$.

Using methods similar to those used in previous sections, we can obtain a meromorphic continuation of this function to $\mathbb{C}^k$. For example, let $r = 2$. We will try to find the value of

$$\zeta^*(s_1, s_2; x_1, x_2)$$

when $s_i = 1 - k_i, k_i \in \mathbb{Z}, k_i \geq 1$.

Then, by Theorem 3.3.2,

$$\begin{aligned} \zeta^*(1 - k_1, 1 - k_2; x_1, x_2) &= \sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1} \binom{k_1 - 1}{j_1}\binom{k_2 - 1}{j_2} x_1^{j_1} x_2^{j_2} \zeta(1 - k_1 + 1 - k_2 + j_1 + j_2) \\ &= -\sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1} \binom{k_1 - 1}{j_1}\binom{k_2 - 1}{j_2} x_1^{j_1} x_2^{j_2} \frac{B_{l(j_1,j_2)}}{l(j_1, j_2)}, \end{aligned}$$

where $l(j_1, j_2) = k_1 + k_2 - 1 - j_1 - j_2$.

More generally, we can show that

$$\zeta(1-k_1, 1-k_2, \ldots, 1-k_r; x_1, x_2, \ldots, x_r)$$

is a polynomial in $x_1, x_2, \ldots, x_r$ with rational coefficients given by Bernoulli numbers. We may also consider these functions for $x_i \in \mathbb{D}$. This gives us a further generalization of (1.1).

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