Phy 352 (Fluid Dynamics) Spring 2013, Final exam (40 points) You are allowed to bring 8 sheets (16 pages) of your own notes to the exam

1. (5 points) Consider the atmosphere of the Sun; suppose you assume it to be isothermal, and in hydrostatic equilibrium, confined by its own gravity. Also assume that the atmosphere consists exclusively of fully ionized hydrogen. The hydrostatic equation for pressure is

$$\frac{dp}{dr} = -\frac{G\,M_{\odot}}{r^2}\,\frac{m_p}{k_B}\,\frac{p}{T}\,,$$

where r is the distance from the center, G is the gravitational constant, $M_{\odot}$ is the mass of the Sun, m_p is the proton mass and k_B is the Boltzmann constant. Assuming further that the temperature varies as

$$T = T_0 \left(\frac{r_0}{r}\right)^{2/7}\,,$$

which satistifes the boundary condition $T = T_0$ at some reference radius $r = r_0$. This gives the solution

$$p = p_0\,\exp\left[\frac{7GM_{\odot}m_p}{5k_B T_0 r_0}\left(\left(\frac{r}{r_0}\right)^{5/7} - 1\right)\right]\,,$$

where p_0 is the pressure at the reference radius r_0. (*I don't want you to work out this solution; its given to you.*) You can consider the Sun to be an isolated body with no other masses around it. Does this solution obey our initial assumption of hydrostatic equilibrium? Why/why not? (2 points) If you think it doesn't, what would the consequence be? (3 points)

Ans: The solution predicts a nonzero pressure at infinity; clearly, this is not hydrostatic equilibrium (2 points). There is a pressure gradient, which will drive a flow away from the Sun; the solar wind (3 points).

2. (10 points) Consider two rigid plates of infinite extent in the x and z direction placed at $y = \pm h$. Consider a 1D laminar flow of a viscous fluid along the x direction under the influence of a constant pressure gradient (in the x direction, of course). How does the velocity u_x vary in the y-dimension? Give an expression, and a qualitative sketch of u as a function of y.

Ans: 1D Poisuelle flow, parabolic velocity profile in the y dimension. The x-momentum equation is (2 points)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Integrating twice, we have (3 points)

$$-\frac{-y^2}{2}\,\frac{dp}{dx} + \eta\,u + Ay + B$$

Applying boundary conditions ($u = 0$ at $y = \pm h$), we get (5 points)

$$u = \frac{y^2}{2\eta}\,\frac{dp}{dx} - \frac{h^2}{2\eta}\,\frac{dp}{dx}$$

(2 points for figuring out the parabolic profile, 1 point for a correct sketch and 2 points for correctly figuring out the boundary conditions)

3. (5 points) In experiments involving turbulence, one usually measures the two point correlation function of velocity fluctuations $R_{ij}(r) = \overline{u_i(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})}$. From this (measured) quantity, one goes on to infer the so-called turbulent "spectrum" $E(k)$ (for Kolmogorov turbulence, $E(k) \propto k^{5/3}$).

   (a) (3 points) Clearly explain the steps involved and state the key assumptions in going from the velocity correlation function to $E(k)$

   (b) (2 points) What is the physical significance of $E(k)$?

Ans: Key point 1: One carries out the (spatial) Fourier transform of the two-point velocity correlation function to obtain $\phi_{ij}(k)$. (2 points) Key point 2: $R_{ij}(r)$ is assumed to be spherically symmetric in real space (assumption of isotropy) so $\phi_{ij}(k)$ is spherically symmetric in k-space (1 point). Therefore (Parseval's theorem) $(1/2)\overline{u^2} = (1/2)\int \phi_{ii}(k)d^3(k) = \int E(k)dk$, which also shows that $E(k)$ can be regarded as the energy spectrum (i.e., the kinetic energy per unit mass per unit interval in k-space) of the turbulence (2 points)

4. (5 points) Consider a strong supersonic shock; the ratio of the upstream (subscript 1) and downstream (subscript 2) densities and velocities are $\rho_2/\rho_1 = u_1/u_2 = \gamma + 1/(\gamma - 1)$.

   (a) (2 points) What is the ratio of the upstream kinetic energy (per unit mass) to the corresponding downstream quantity? Which is larger?

   (b) (3 points) Where does the difference in kinetic energies (per unit mass) "go"?.

   Ans: Energy conservation across a shock gives $(1/2)v^2 + (\gamma/\gamma - 1)p/\rho =$ constant. i.e., the sum of enthalpy and kinetic energy (per unit mass) is constant. The upstream kinetic energy is certainly more than the downstream one (2 points). The difference is accounted for by the increase in enthalpy $(\gamma/\gamma - 1)p/\rho$; in other words, part of the upstream kinetic energy is converted into internal energy (increase in temperature) downstream (3 points)

5. (5 points) True or false? A smooth, solid sphere moving slowly through an inviscid fluid will experience a drag. Explain your answer.

   Ans: False. If the fluid is perfectly inviscid, it slips freely past the sphere and there will be no drag. In reality, however, a real fluid will have some small, nonzero viscosity, and although the bulk flow might be inviscid, viscous effects will assume importance in a thin boundary layer near the body, where velocity gradients are large. There will be some "sticking" at the surface, which will lead to drag.

6. (10 points) Consider 1D, steady state, trans-sonic flow of a compressible fluid flowing through a horizontal pipe with varying cross-sectional area. The continuity equation is

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{u}\frac{du}{dx} + \frac{1}{A}\frac{dA}{dx} = 0$$

   and the momentum equation is

$$\rho\, u\frac{du}{dx} + \frac{dp}{dx} = 0$$

   We know that the flow can pass through a sonic point where $dA/dx = 0$. Obtain an equation that will yield the value of du/dx at the sonic point as a function of the speed of sound, A, and derivatives of A (at the sonic point). You can assume the speed of sound to be constant with x. Note, your equation should yield a specific numerical value for du/dx if all the quantities are specified.

   Ans: (3 points)

$$\frac{du}{dx} = \frac{(u/A)dA/dx}{(u^2/c_s^2) - 1}$$

   . Letting $u \to c_s$ and applying the L' Hospital rule we get (4 points)

$$2(\frac{du}{dx})^2 - \frac{c_s}{A}\frac{dA}{dx}\frac{du}{dx} - \frac{c_s^2}{A}\frac{d^2A}{dx^2} + \frac{c_s^2}{A^2}(\frac{dA}{dx})^2 = 0$$

   . Since $dA/dx = 0$, this reduces to (3 points)

$$2(\frac{du}{dx})^2 - \frac{c_s^2}{A}\frac{d^2A}{dx^2} = 0$$