Runge $2^{nd}$ Order Method

Major: All Engineering Majors

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Modified by P. Goel for IDC 103

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Transforming Numerical Methods Education for STEM Undergraduates
Runge-Kutta 2\textsuperscript{nd} Order Method

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Heun’s Method

Heun’s method

Here \( a_2 = 1/2 \) is chosen resulting in

\[
y_{i+1} = y_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h
\]

where

\[
k_1 = f(x_i, y_i)
\]

\[
k_2 = f(x_i + h, y_i + k_1 h)
\]

\[\text{Slope} = f(x_i, y_i)\]

\[\text{Average Slope} = \frac{1}{2} [f(x_i + h, y_i + k_1 h) + f(x_i, y_i)]\]

Figure 1 Runge-Kutta 2nd order method (Heun’s method)
Runge-Kutta 2nd Order Method

For \( \frac{dy}{dx} = f(x, y) \), \( y(0) = y_0 \)

Runge Kutta 2nd order method is given by

\[
y_{i+1} = y_i + \left( a_1 k_1 + a_2 k_2 \right)h
\]

where

\[
k_1 = f(x_i, y_i)\quad \frac{1}{2}
\]

\[
k_2 = f(x_i + p_1h, y_i + q_1k_1h)\quad \frac{1}{2}
\]

for Heun's Method

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Midpoint Method

Here \( a_2 = 1 \) is chosen, giving

\[
a_1 = 0 \\
p_1 = \frac{1}{2} \\
q_{11} = \frac{1}{2}
\]

resulting in

\[
y_{i+1} = y_i + k_2 h
\]

where

\[
k_1 = f(x_i, y_i) \\
k_2 = f\left(x_i + \frac{1}{2} h, y_i + \frac{1}{2} k_1 h\right)
\]
Ralston’s Method

Here \( a_2 = \frac{2}{3} \) is chosen, giving

\[
\begin{align*}
  a_1 &= \frac{1}{3} \\
  p_1 &= \frac{3}{4} \\
  q_{11} &= \frac{3}{4}
\end{align*}
\]

resulting in

\[
y_{i+1} = y_i + \left( \frac{1}{3} k_1 + \frac{2}{3} k_2 \right) h
\]

where

\[
\begin{align*}
  k_1 &= f(x_i, y_i) \\
  k_2 &= f\left( x_i + \frac{3}{4} h, y_i + \frac{3}{4} k_1 h \right)
\end{align*}
\]
How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

\[ \frac{dy}{dx} = f(x, y) \]

Example

\[ \frac{dy}{dx} + 2y = 1.3e^{-x}, \quad y(0) = 5 \]

is rewritten as

\[ \frac{dy}{dx} = 1.3e^{-x} - 2y, \quad y(0) = 5 \]

In this case

\[ f(x, y) = 1.3e^{-x} - 2y \]
Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

\[
\frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right), \theta(0) = 1200K
\]

Find the temperature at \( t = 480 \) seconds using Heun’s method. Assume a step size of \( h = 240 \) seconds.

\[
\frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right)
\]

\[
f(t, \theta) = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right)
\]

\[
\theta_{i+1} = \theta_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h
\]
Solution

Step 1: \( i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200 K \)

\[
k_1 = f(t_0, \theta_0) \\
= f(0,1200) \\
= -2.2067 \times 10^{-12} \left(1200^4 - 81 \times 10^8\right) \\
= -4.5579
\]

\[
k_2 = f(t_0 + h, \theta_0 + k_1 h) \\
= f(0 + 240,1200 + (-4.5579)240) \\
= f(240,106.09) \\
= -2.2067 \times 10^{-12} \left(106.09^4 - 81 \times 10^8\right) \\
= 0.017595
\]

\[
\theta_1 = \theta_0 + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \\
= 1200 + \left(\frac{1}{2} (-4.5579) + \frac{1}{2} (0.017595)\right)240 \\
= 1200 + (-2.2702)240 \\
= 655.16 K
\]
Solution Cont

**Step 2:** \( i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K \)

\[
k_1 = f(t_1, \theta_1) = f(240, 655.16)
\]

\[
= -2.2067 \times 10^{-12} \left( 655.16^4 - 81 \times 10^8 \right)
\]

\[
= -0.38869
\]

\[
k_2 = f(t_1 + h, \theta_1 + k_1h) = f(240 + 240, 655.16 + (-0.38869)240)
\]

\[
= f(480.561.87)
\]

\[
= -2.2067 \times 10^{-12} \left( 561.87^4 - 81 \times 10^8 \right)
\]

\[
= -0.20206
\]

\[
\theta_2 = \theta_1 + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right)h
\]

\[
= 655.16 + \left( \frac{1}{2} (-0.38869) + \frac{1}{2} (-0.20206) \right)240
\]

\[
= 655.16 + (-0.29538)240
\]

\[
= 584.27K
\]
The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

\begin{align*}
0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333 \theta) &= -0.22067 \times 10^{-3} t - 2.9282
\end{align*}

The solution to this nonlinear equation at \( t=480 \) seconds is

\( \theta(480) = 647.57 K \)
Comparison with exact results

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Heun’s method results for different step sizes}
\end{figure}

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**Effect of step size**

**Table 1. Temperature at 480 seconds as a function of step size, h**

| Step size, $h$ | $\theta(480)$ | $E_t$    | $|\varepsilon_t|\%$ |
|---------------|---------------|----------|------------------|
| 480           | $-393.87$     | 1041.4   | 160.82           |
| 240           | $584.27$      | 63.304   | 9.7756           |
| 120           | $651.35$      | $-3.7762$| 0.58313          |
| 60            | $649.91$      | $-2.3406$| 0.36145          |
| 30            | $648.21$      | $-0.63219$| 0.097625        |

$\theta(480) = 647.57K \quad \text{(exact)}$
Effects of step size on Heun’s Method

Figure 3. Effect of step size in Heun’s method
Comparison of Euler and Runge-Kutta 2\textsuperscript{nd} Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

<table>
<thead>
<tr>
<th>Step size, (h)</th>
<th>(\theta(480))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Euler</td>
</tr>
<tr>
<td>480</td>
<td>-987.84</td>
</tr>
<tr>
<td>240</td>
<td>110.32</td>
</tr>
<tr>
<td>120</td>
<td>546.77</td>
</tr>
<tr>
<td>60</td>
<td>614.97</td>
</tr>
<tr>
<td>30</td>
<td>632.77</td>
</tr>
</tbody>
</table>

\[ \theta(480) = 647.57K \quad \text{(exact)} \]
Comparison of Euler and Runge-Kutta 2\textsuperscript{nd} Order Methods

\textbf{Table 2.} Comparison of Euler and the Runge-Kutta methods

| Step size, \( h \) | \( |\varepsilon_i|\% \) | \( \text{Euler} \) | \( \text{Heun} \) | \( \text{Midpoint} \) | \( \text{Ralston} \) |
|-------------------|-----------------|---------------|----------------|-----------------|-----------------|
| 480               |                 | 252.54        | 160.82         | 86.612          | 30.544          |
| 240               |                 | 82.964        | 9.7756         | 50.851          | 6.5537          |
| 120               |                 | 15.566        | 0.58313        | 6.5823          | 3.1092          |
| 60                |                 | 5.0352        | 0.36145        | 1.1239          | 0.72299         |
| 30                |                 | 2.2864        | 0.097625       | 0.22353         | 0.15940         |

\[ \theta(480) = 647.57 K \quad (\text{exact}) \]
Comparison of Euler and Runge-Kutta 2\textsuperscript{nd} Order Methods

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Comparison of Euler and Runge Kutta 2\textsuperscript{nd} order methods with exact results.}
\end{figure}

\begin{center}
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\end{center}
Is there a method to this madness?!

First of all note the Taylor expansion in two variables is:

\[
\begin{aligned}
    f(x+h, y+k) &= f(x, y) + hf_x + kf_y \\
    &+ \frac{1}{2!} \left( h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} \right) \\
    &+ \frac{1}{3!} \left( h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy} \right) \\
    &+ \ldots
\end{aligned}
\]
First

\[ y_{i+1} = y_i + \frac{dy}{dx} \bigg|_{x_i, y_i} h + \frac{1}{2} \frac{d^2y}{dx^2} \bigg|_{x_i, y_i} h^2 + o(h^3) \]

Note that

\[ f' (x, y) = f_x + f_y \frac{dy}{dx} \]

thus

\[ y_{i+1} = y_i + f_x h + \frac{1}{2} f_x h^2 + \frac{1}{2} f_y f_x h^2 + o(h^3) \]
Next, RK Methods have the form

\[ y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h \]

Notice that \( k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \)

\[ = f + p_1 hf_x + q_{11} k_1 hf_y + O(h^2) \]

(by Taylor expansion)

So

\[ y_{i+1} = y_i + \left[ a_1 f + a_2 \left( f + p_1 hf_x + q_{11} k_1 hf_y + O(h^2) \right) \right] h \]

\[ = y_i + (a_1 + a_2) hf + a_2 p_1 h^2 f_x + a_2 q_{11} h^2 f_y + O(h^3) \]
Comparing (1) & (2) we have:

\[ a_1 + a_2 = 1 \]
\[ a_2 \, b_1 = \frac{1}{2} \]
\[ a_2 \, q_{11} = \frac{1}{2} \]

So

<table>
<thead>
<tr>
<th></th>
<th>Henn</th>
<th>Midpt</th>
<th>Ralston</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_2</td>
<td>1/2</td>
<td></td>
<td>2/3</td>
</tr>
<tr>
<td>a_1</td>
<td>1/2</td>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>p_1</td>
<td></td>
<td>1/2</td>
<td>3/4</td>
</tr>
<tr>
<td>q_{11}</td>
<td>1</td>
<td>1/2</td>
<td>3/4</td>
</tr>
</tbody>
</table>
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html
THE END

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