Arithmetic aspects of locally symmetric spaces.

Supriya Pisolkar (IISER - Pune)

IWM-University of Hyderabad-2016
Central theme of this talk

Let $M$ be a compact connected Riemannian manifold. In differential geometry, one associates to $M$ the following data:

- $\varepsilon(M) :=$ the spectrum of $M$, i.e. the set of nonzero eigenvalues counted with multiplicities of the Laplace-Beltrami operator $\Delta_M$ acting on the space of smooth functions on $M$.
- $L(M) :=$ the (weak) length spectrum of $M$, i.e. the set of lengths of closed geodesics (without multiplicities).

Question (Inverse spectral problem) To what extend these spectra determine the manifold? Most of the results in this direction are negative!
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Isospectral Riemannian manifolds

Two compact, connected Riemannian manifolds $M_1$ and $M_2$ are said to be isospectral if $\varepsilon(M_1) = \varepsilon(M_2)$.

Example
Any two isometric Riemannian manifolds are isospectral.

Similarly, $M_1$ and $M_2$ are called iso-length spectral if $L(M_1) = L(M_2)$.

Theorem (Gangoli (for rank 1), Duistermaat-Kolk-Varadarajan (for general case))
Any two compact isospectral locally symmetric spaces are iso-length spectral.
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**Question (Weaker form of Kac’s question)**

Are any two isospectral compact Riemannian manifolds necessarily commensurable?

This has been extensively studied in the context of locally symmetric spaces.
Locally symmetric spaces

Locally symmetric spaces are the spaces of the form $\Gamma \backslash G/K$ where $G$ is a real semisimple Lie group, $K$ is a maximal compact subgroup of $G$ and $\Gamma$ is a lattice in $G$. The universal cover $G/K$ carries a natural $G$-invariant metric coming from the Killing form on the Lie algebra $\mathfrak{g}$ of $G$. 

Example

When $G = \text{SL}(2, \mathbb{R})$ and $K = \text{SO}(2, \mathbb{R})$ then, $G/K \cong \mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}; y > 0 \}$. For $N > 3$, consider the $\Gamma(N)$ then, $\Gamma(N) \backslash \text{SL}(2, \mathbb{R})/\text{SO}(2, \mathbb{R})$ is a locally symmetric space.
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Isospectrality $\Rightarrow$ Commensurability?

Lubotzky-Samuels-Vishne '2006 by using the Langlands correspondence, have constructed non-commensurable but isospectral locally symmetric spaces associated with absolutely simple real groups of type $A_n$. Nevertheless, the answer to above question is in the affirmative for several classes of arithmetically defined locally symmetric spaces.

A. Reid '1992 - If $M_1$ and $M_2$ are two isospectral nonisometric arithmetic hyperbolic 2-manifolds then $M_1$ and $M_2$ are commensurable.

Chinburg-Hamilton-Long and Reid '2008 - Similar result for arithmetically defined hyperbolic 3-manifolds.

Gopal Prasad and Rapinchuk '2009 - Assuming the validity of Schanuel's conjecture on transcendental numbers, any two compact isospectral arithmetically defined locally symmetric spaces associated with the absolutely simple real algebraic groups of type other than $A_n$, $D_{2n+1}$ ($n > 1$) or $E_6$ are necessarily commensurable.
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One of the important ingredients of their proof is the connection between Laplace spectrum $\epsilon(M)$ and length spectrum $L(M)$. They first observed that isospectral spaces under consideration are length-commensurable, which means $L(M_1) = Q L(M_2)$ (length commensurability is weaker than the notion of iso-length spectral). Isospectrality $\downarrow \downarrow$ Length commensurability $\downarrow \downarrow$ Commensurability To prove that length commensurability implies commensurability they have introduced a new notion of weak commensurability of Zariski dense subgroups in absolutely almost simple groups.
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To prove that length commensurability implies commensurability they have introduced a new notion of weak commensurability of Zariski dense subgroups in absolutely almost simple groups.
Weak commensurability

**Definition**

For $i = 1, 2$, $G_i$ - semisimple group defined over a field $F (\text{char} = 0)$. $\Gamma_i \subset G_i(F)$ - Zariski dense subgroup.

Then $\Gamma_1$ and $\Gamma_2$ are said to be **weakly commensurable** if given any element of infinite order $\gamma_1 \in \Gamma_1$ there exists an element of infinite order $\gamma_2 \in \Gamma_2$ such that the subgroup of $\overline{F}^{\times}$ generated by eigenvalues of $\gamma_1$ (resp. $\gamma_2$) (in a faithful representation of $G_1$) intersect nontrivially the subgroup generated by the eigenvalues of an element $\gamma_2$ ( resp. $\gamma_1$).
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Schanuel’s conjecture If $z_1, z_2, \cdots, z_n$ are $\mathbb{Q}$-linearly independent complex numbers, then the transcendence degree over $\mathbb{Q}$ of the field generated by

$$z_1, z_2, \cdots, z_n, e^{z_1}, \cdots, e^{z_n}$$

is at least $n$. 

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Theorem (G. Prasad and Rapinchuk)

Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero containing weakly commensurable Zariski dense arithmetic subgroups. Assume the validity of Schanuel's conjecture. Then,

1. The groups $G_1$ and $G_2$ are either of the same geometric type or one of them is of type $B_n$ and other is of type $C_n$ for some $n \geq 3$. 

2. If the groups are of the same type different from $A_n$, $D_{2n} + 1$ ($n > 1$) or $E_6$ then the lattices are commensurable.

3. In any weakly commensurable class of arithmetic lattices, there are only finitely many commensurability classes of arithmetic lattices.
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Summary

Isospectrality

↓

Commensurability
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Isospectrality $\xrightarrow{\text{DKV}}$ Length commensurability

Commensurability $\xleftarrow{\text{PR}}$ Weak commensurability

↓

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↓

Schanuel’s conj.
Can we avoid Schanuel’s conjecture?

Inspired by the results of Parsad and Rapinchuk we asked whether Schanuel’s conjecture can be avoided under the stronger hypothesis that the Zariski dense subgroups $\Gamma_1$ and $\Gamma_2$ defining the locally symmetric spaces are representation equivalent lattices?

Advantage of working with representation equivalence is that one can work with arithmetic setting when there is no Riemannian geometric structure on the given space.

We introduce a new relation characteristic equivalence on the class of arithmetic lattices, stronger than weak commensurability. This simplifies some of the arguments used by Prasad and Rapinchuk to deduce commensurability type of results.
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Representation equivalence

Let $\Gamma$ be a locally compact, unimodular topological group. $\Gamma$ is a uniform lattice in $G$. Let $R_{\Gamma}$ denote the right regular representation of $G$ on the space $L^2(\Gamma \backslash G)$ given by:

$$(R_{\Gamma}(g)f)(x) = f(xg), \quad f \in L^2(\Gamma \backslash G), \quad g, x \in G.$$ 

As a $G$-space, $L^2(\Gamma \backslash G)$ breaks up as a direct sum of irreducible unitary representations of $G$, $L^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \pi$. 

**Definition:** Two uniform lattices $\Gamma_1$ and $\Gamma_2$ inside a locally compact group $G$ are said to be representation equivalent if $L^2(\Gamma_1 \backslash G) \cong L^2(\Gamma_2 \backslash G)$ as $G$-spaces. 

DeTurck-Gordon: Let $G$ be a locally compact topological group which acts on a Riemannian manifold $M$. If $\Gamma_1, \Gamma_2 \subset G$ are two representation equivalent lattices which act properly discontinuously and freely on $M$, then $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are isospectral for the Laplacian acting on the space of smooth functions.
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Representation equivalence \( \iff \) commensurability

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$RE \implies$ Isospectrality $\implies$ Length commensurability

Commensurability $\iff$ Weak commensurability

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Let $P(Ad_G(\gamma), x)$ denote the characteristic polynomial of $Ad(\gamma)$. 

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Arithmetic aspects of locally symmetric space
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**Definition**

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$\Gamma_i \subset G_i(K)$ - arithmetic subgroups. Then $\Gamma_1$ and $\Gamma_2$ are called Characteristically equivalent if given any element $\gamma \in \Gamma_1$ there exists an element $\gamma_2 \in \Gamma_2$ such that $P(Ad_{G_1}(\gamma_1), X) = P(Ad_{G_2}(\gamma_2), X)$. 
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By using trace formula for compact quotients we have proved that

Proposition

If $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \in G_2(K)$ are representation equivalent lattices then
they are characteristically equivalent.
Theorem ( -, Rajan, Bhagwat '2015)

For \( i = 1, 2 \) let,
\[ G_i - \text{connected, absolutely almost simple anisotropic algebraic groups defined over a number field } K, \]
\[ \Gamma_i \subset G_i(K) - \text{Characteristically equivalent arithmetic subgroups}. \]
Then,

1. The groups \( G_1 \) and \( G_2 \) are either of the same geometric type or one of them is of type \( B_n \) and other is of type \( C_n \) for some \( n \geq 3 \).
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\[ \text{RE} \rightarrow \text{Isospectrality} \rightarrow \text{Length commensurability} \]
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In the sequel to their paper on weak commnensurability, Prasad and Rapinchuk analyzed the situation where locally symmetric spaces $M_1$ and $M_2$ are not length commensurable. They studied the question how different the sets $\mathbb{Q} \cdot L(M_1)$ and $\mathbb{Q} \cdot L(M_2)$ can be, and can $L(M_1)$ and $L(M_2)$ be related by any reasonable way? To answer this, they have studied the field $\mathbb{F}(M)$ which is a subfield of $\mathbb{R}$ generated by the set $L(M)$, for a Riemannian manifold $M$. By assuming the validity of Schanuel's conjecture, they proved that these fields determine the commensurability class. Their precise result is,
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**Theorem**

Let $M_i = X_i/\Gamma_i$ be the locally symmetric space, where $X_i$ is a symmetric space corresponding to the absolutely simple real algebraic groups $G_i$, for $i = 1, 2$. If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are not length commensurable then either $F_1 F_2 / F_1$ or $F_1 F_2 / F_2$ is of infinite transcendence degree.
Splitting fields of the characteristic polynomials

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Notation:
\( G \) - a connected split algebraic group defined over a number field \( K \).
Fix an algebraic closure \( \overline{K} \) of \( K \).
\( T \) - a maximal torus in \( G \) defined over \( K \).
\( \Phi = \Phi(G, T) \subset X^*(T) \) the corresponding root system.
Then the Galois group \( G_K \) preserves the root system \( \Phi \) inducing the homomorphism
\[ \theta_T : G_K \to Aut(\Phi(G, T)) \]
Since \( G \) is split, the image lands in the Weyl group \( W(G, T) \).
Definition

A semisimple regular element $g \in G(K)$ is said to be generic $K$-regular if the torus $T_g$ which is a connected component of the centraliser $Z(g)$ in $G$ satisfies the property that the image $\theta_{T_g}(G_K) = W(G, T)$.

It is known that a finitely generated Zariski dense subgroup $\Gamma \subset G(K)$ contains a generic $K$-regular element of infinite order.

Let, $K_g$ - the minimal splitting field of the corresponding torus $T_g$, $K(g, \text{Ad})$ - the splitting field of the characteristic polynomial $P(g, \text{Ad})$ of the linear transformation $\text{Ad}(g)$ then $K_g = K(g, \text{Ad})$. 

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Main theorem

Assumptions: (1) $G_1$ and $G_2$ - connected, split, absolutely almost simple algebraic groups defined over a number field $K$.
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Call the pair $(G_1, G_2)$ exceptional if one of the following hold:
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(3) Suppose $\mathcal{F}_i = \mathcal{F}(\Gamma_i, K)$ be the subfield of $\overline{K}$ given by the composite of the fields $K_{\gamma}$ where $\gamma \in \Gamma_i$ varies over the set of generic $K$-regular elements in $\Gamma_i$. 


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Theorem ( -, C. S. Rajan '2016)

Assume that $G_1$ and $G_2$ are as above and they do not form an exceptional pair, then the compositum $F_1F_2$ is of infinite degree over either $F_1$ or $F_2$. 
Open Problems

- It is not known how to avoid Schanuel’s conjecture in the geometric result of Prasad and Rapinchuk that connects length commensurability to weak commensurability.

- The seemingly weak notion of weak commensurability has many useful applications. For example, suppose, for \( i = 1, 2 \), \( G_i \) is an absolutely almost simple algebraic group over a non-archimedean local field \( F \), then it is known that if the two Zariski dense subgroups \( \Gamma_1 \) and \( \Gamma_2 \) in \( G_1(F) \) and \( G_2(F) \) are weakly commensurable and one of them is discrete then the other is also discrete. But it is not known whether the same is true if \( F = R \) or \( C \).

- Another interesting question is whether weak commensurability preserves cocompactness of the lattices \( \Gamma_i \subset G_i(F) \) when \( F = R \) or \( C \). (In joint work with C. Bhagwat ‘2016 this question is settled in the affirmative for representation equivalent lattices)
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Thank You!