Cantor Set

Cantor set is a special subset of the closed interval $[0,1]$ invented by a German mathematician Georg Cantor in 1883. We have already discussed the construction of this ‘ternary’ set in the class but let me quickly recall it.

Let $I_0 := [0,1]$. Remove the open middle third $(1/3,2/3)$ from $I_0$.
Let $I_1 := [0,1/3] \cup [2/3,1]$, precisely $I_1 = I_0 \setminus (1/3,2/3)$.
Remove middle third open intervals $(1/9,2/9)$ and $(7/9,8/9)$ from the respective closed intervals whose union is $I_1$.
Let $I_2 := [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,9/9]$ say, $I_2^1 \cup I_2^2 \cup I_2^3 \cup I_2^4$.

Continue this procedure of removal of open intervals from each closed interval $I_k^j$, $1 \leq k \leq 2^j$ whose union is $I_j$, $j = 3,4,\cdots$ and get $I_{j+1}$ by taking the union of whatever is left from each $I_k^j$ after the removal of open intervals as before.

Define the Cantor set $\mathcal{C} := \cap_{n=1}^{\infty} I_n$.

Observations/Facts about $\mathcal{C}$

(1) Cantor is non-empty: Clearly all end points of the closed intervals comprising $I_n$ for every $n = 1,2,\cdots$ are in $\mathcal{C}$. Further it contains countably many points, for example points of the form $1/3^n$, $n = 1,2,3,\cdots$.

(2) Cantor set is a closed subset in $[0,1]$ : Each $I_n$ is a union of $2^n$ closed intervals. As a finite union of closed intervals, it is a closed set in $[0,1]$ with the usual metric. Then $\mathcal{C}$ is a closed subset of $[0,1]$ being the intersection of closed sets $I_n$, $n = 1,2,\cdots$.

(3) Cantor set has length 0 : Each $I_n$ is a union of $2^n$ closed intervals, each of length $1/3^n$. Thus, length of $I_1$ i.e. $\ell(I_1) = 2 \times 1/3 = 2/3$, $\ell(I_2) = 4 \times (1/9) = (2/3)^2 \cdots \ell(I_n) = 2^n \times (1/3^n) = (2/3)^n$. This implies that the length of the Cantor set $\ell(\mathcal{C}) = \lim_{n \to \infty} (2/3)^n = 0$.

(4) Cantor set is an uncountable set: Suppose on contrary $\mathcal{C}$ is a countable set. Let $x \in \mathcal{C}$. Thus $x \in I_n$, $n = 1,2,\cdots$. Note that every closed interval $I_j^k$ in $I_j$ gives rise to two closed intervals $L$ and $R$ in the next step of forming $I_{j+1}$. As $x \in \mathcal{C} \subset [0,1]$, $x \in L$ or $R$ in $I_1$. Suppose it is in $L$ in $I_1$. Then $x$ can either be in $L$ or in $R$ in $I_2$. Suppose it is $R$ in $I_2$ and then in $R$ in $I_3$. Tracing down the intervals in which $x$ lies in each $I_n$, we get a unique infinite sequence of $L$’s and $R$’s, which in this case is $LRR\cdots$.

Therefore, given a $x \in \mathcal{C}$ we get a unique sequence of $L$’s and $R$’s.
Now, suppose we consider this same sequence $LRR \cdots$ as above and go on choosing random points $y_n$ at every stage in $I_n$ from $L$ or $R$ as given in the $n^{th}$ place of the sequence. Then it is easy to check that $\{y_n\}_n$ forms a Cauchy sequence and as $n \to \infty$ it converges to the point $x$ as above.

Let us re-name $L \leftrightarrow 0$ and $R \leftrightarrow 2$. Then it will give a ternary expansion for every $x \in \mathbb{C}$. For example in the above case, $x \leftrightarrow LRR \cdots \leftrightarrow 022 \cdots$ (or to be precise $0.022 \cdots$).

As $\mathbb{C}$ is assumed to be a countable set we can list its elements and write their respective ternary expansion in $0's$ and $2's$.

\[
a_1 = a_{11}a_{12}a_{13} \cdots \\
a_2 = a_{21}a_{22}a_{23} \cdots \\
a_3 = a_{31}a_{32}a_{33} \cdots \\
\vdots \\
a_n = a_{n1}a_{n2}a_{n3} \cdots \\
\vdots
\]

We will now use Cantor’s diagonal argument to construct an element $b \in \mathbb{C}$ which is not in the list $\{a_1, a_2, a_3, \cdots\}$. Let $b = b_1b_2b_3 \cdots$ where $b_i = 0$ if $a_{ii} = 2$ and $b_i = 2$ if $a_{ii} = 0$. We can check that at every $n^{th}$ stage, $b_n$ differs from $a_n$. As $b$ also has a ternary expression in $0's$ and $2's$ it belongs to $\mathbb{C}$ (For this use the fact (8)). Thus proving that $\mathbb{C}$ is an uncountable set.

(5) Cantor set has no intervals: (part of tutorial)

(6) Cantor set is a no where dense subset of $[0, 1]$ (part of tutorial)

(7) Cantor set is a totally disconnected set. It is also compact in the usual topology on $[0, 1]$ (Later)

**Ternary expansions and the Cantor Set**

(8) An element $x$ in $[0, 1]$ belongs to the Cantor set $\mathbb{C}$ if and only if $x$ has only $0's$ and $2's$ in it’s ternary expansion:

Note that a real number $x$ may not have a unique ternary expansion (for example $1/3 = (0.1)_3$. Also, $(0.0\overline{2}2)_3 = 2/3^2 + 2/3^3 + 2/3^4 + \cdots = 2/3^2(1 + 1/3 + 1/3^2 + \cdots) = 1/3$). But $x$ can have at most one ternary expansion consisting of only $0's$ and $2's$ (Prove this fact as an exercise).

*Proof of (8).* Now, let $x \in \mathbb{C}$. Then following the argument of (4), we know that $x$ has a unique ternary expansion in $0's$ and $2's$. Now let $x \in [0, 1]$ such that it has only $0's$ and $2's$ in it’s ternary expansion i.e. $x = \sum_{k=1}^{\infty} a_k3^{-k}$, where $a_k = 0$ or $2$. 


We will show by using induction on $k$ that $x \in I_k$ for every $k$ and thus is contained in $C$. Clearly $x \in I_0 = [0, 1]$. Next if $a_1 = 0$ then $x \leq (0.1)_3 = 1/3$; if $a_1 = 2$ then $x \geq (0.2)_3 = 2/3$. So $x \in [0, 1/3] \cup [2/3, 1] = I_1$. This gives base case for induction.

Assume that $x \in I_k$. Then $x$ is in one of the $2^k$ disjoint closed intervals comprising $I_k$, say $[a, b]$. The construction process of removing middle third open intervals ensures that every left end point in $I_k$ must either be a left end point of $I_{k-1}$ or is a sum of a left end point of $I_{k-1}$ and $2/3^k$. Now removing open middle third set from $[a, b]$ results in two disjoint closed intervals $L = [a, a + 1/3^k]$ and a right closed interval $[b - 1/3^{k+1}, b]$. If $a_{k+1} = 0$ then $x \leq a + 1/3^{k+1}$ and is thus contained in $L$. If $a_{k+1} = 2$, then $x \geq a + 2/3^{k+1}$ and is contained in the right interval $R$. Either way $x \in I_{k+1}$. So the inductive step is complete. Thus $x \in C$. □