Composite Fermions And The Fractional Quantum Hall Effect: A Tutorial

Exercises

i. Confirm the commutation relations for the operators $a$, $b$, $a^\dagger$ and $b^\dagger$.

ii. Confirm that the eigenstates for magnetic field pointing in the $-z$ direction are complex conjugates of the above wave functions.

iii. Obtain the second Landau level wave function $|1, m\rangle$.

iv. Show that the polynomial part of the single particle wave function (i.e. the factor multiplying the gaussian) of the $n$th Landau level involve at most $n$ powers of $\bar{z}$.

v. Derive

$$\Phi_1 = \prod_{j<k}(z_j - z_k) \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right]. \quad (1)$$

vi. Show that the wave function of a lowest filled Landau level with a hole in the $m = 0$ state is given by

$$\phi_1^{\text{hole}} = \left( \prod_j z_j \right) \Phi_1 \quad (2)$$

vii. Show that the wave function of a lowest filled Landau level with an additional electron in the second LL in the $m = -1$ state (smallest angular momentum in the second Landau level) is given by

$$\phi_1^{\text{particle}} = \sum_{i=1}^{N} \left[ \prod_j (z_i - z_j)^{-1} \right] \bar{z}_i \Phi_1 \quad (3)$$

where the prime denotes the condition $j \neq i$.

viii. Show that the wave function for two filled LLs is given by

$$\Phi_2 = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 & \cdots \\ \ddots & \ddots & \ddots & \cdots \\ z_1^{N/2-1} & z_2^{N/2-1} & z_3^{N/2-1} & \cdots \\ \bar{z}_1 z_1 & \bar{z}_2 z_2 & \bar{z}_3 z_3 & \cdots \\ \ddots & \ddots & \ddots & \cdots \\ \bar{z}_1 z_1^{N/2-1} & \bar{z}_2 z_2^{N/2-1} & \bar{z}_3 z_3^{N/2-1} & \cdots \\ \end{bmatrix} \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right]. \quad (4)$$

ix. Show that

$$\phi_\eta(r) = \frac{1}{\sqrt{2\pi}} \exp \left[ \frac{1}{2} \eta z - \frac{1}{4} |z|^2 - \frac{1}{4} |\eta|^2 \right] \quad (5)$$

is a coherent state, i.e. is an eigenstate of the angular momentum lowering operator $b$. Show that it represents a wave packet localized at $\eta$. 
x. Obtain the solution for an electron in a parabolic confinement potential in the presence of a magnetic field

\[ H = \frac{1}{2m_b} \left( p + \frac{e}{c} A \right)^2 + \frac{1}{2} m_b \omega_0^2 (x^2 + y^2) \]  

(6)

where \( \omega_0 \) is a measure of the strength of the confinement. The solutions are known as Fock-Darwin levels.

xi. Evaluate the Berry phase for the localized wave packet given in Eq. 5 for a circular loop and show that it is equal to the Aharonov-Bohm phase.

xii. Show that the ground state wave function of one filled \( \Lambda \) level of composite fermions \( (\nu^* = 1) \) is identical to Laughlin’s wave function

\[ \Psi_{1/m} = \prod_{j<k} (z_j - z_k)^m \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right] \]  

(7)

at \( \nu = 1/(2p + 1) \). No lowest Landau level projection is needed in this case.

xiii. We ask how a single particle orbital is modified by adiabatic insertion of a point flux tube. Obtain the solution for the Hamiltonian

\[ H(\alpha) = \frac{1}{2m_b} \left( p + \frac{e}{c} A + \frac{e}{c} a_\alpha \right)^2 \]  

(8)

where

\[ a_\alpha = \frac{\alpha}{2\pi} \phi_0 \nabla \theta \]  

(9)

produces a flux of strength \( \alpha \phi_0 \) at the origin. (Hint: This can be solved by a gauge transformation, while making sure that it is the physical wave function is single valued.) Then ask how an orbital evolves as we increase \( \alpha \) from \( 0 \) to \( 1 \).

xiv. We derived the wave functions for the hole and particle of \( \nu = 1 \) state. Construct the corresponding wave functions for the CF-quasihole and the CF-quasiparticle at \( \nu^* = 1 \). Then go ahead and construct the wave functions for two CF-quasiholes and two CF-quasiparticles in the innermost angular momentum states.

xv. Prove

\[ \mathcal{P}_{LLL} e^{-\frac{i}{2} z^2 z^m z^s} = e^{-\frac{i}{2} \hat{z} z^m} \left( \frac{\partial}{\partial z} z^s \right)^m \]  

(10)

(where \( \mathcal{P}_{LLL} \) is the lowest Landau level projection operator) by two methods. (i) Noting that the projection is nonzero only for the lowest Landau level orbital with angular momentum \( s - m \), evaluate its coefficient by using completeness relation. (ii) Consider \( \bar{z}^m \phi \), where \( \phi \) is an arbitrary lowest Landau level wave function. Express \( \bar{z} \) in terms of ladder operators and show that the lowest Landau level projection produces \( (\sqrt{2}b)^m \phi \).

xvi. Prove that for any operator \( V(z, \bar{z}) \), \( V_p \) defined as

\[ V_p(\bar{z}, z) = :V \left( \bar{z} \rightarrow 2 \frac{\partial}{\partial z}, z \right): \]  

(11)

has the same matrix elements within the lowest Landau level. Here the normal ordering symbol : : indicates that we bring all \( \bar{z} \)'s to the left of the \( z \)'s, and then make the replacement \( \bar{z} \rightarrow 2\partial/\partial z \) with the understanding that the derivatives do not act on the gaussian factor. \( V_p \) defines the lowest Landau level projection of \( V \), because when applied to a lowest Landau level state, it causes no mixing with higher Landau level states.

xvii. Show that \( x_p \) and \( y_p \), the projected coordinates, obey the commutator

\[ [x_p, y_p] = it^2 \]  

(12)

The lowest Landau level space is said to be non-commutative.

xviii. Obtain the relative braid statistics of a quasiparticle going around a quasihole. Assuming that an exciton produces no statistical phase, deduce the braid statistics for quasiholes. Finally, show that a bound state of \( 2p_n \pm 1 \) quasiparticles has the same charge and braid statistics as an electron.